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## Some topics in reliability theory

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Some topics in reliability  
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by

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DEDICATION

In The Name of

God

To

My Mother

## PROLOGUE

This thesis consists of two parts. In Chapter 2 and Chapter 3 we have studied various concepts of negative dependence in multivariate case, while in Chapter 4 we have studied two important classes of multivariate life distributions useful in reliability theory.



## 1. INTRODUCTION

### 1.1. Background

The study of the nature of dependence among two or more random variables is important both from the probabilistic and statistical viewpoint. Perhaps the first effort to gain general results for dependence was by Lehmann (1966). According to his definition, two random variables  $X_1$  and  $X_2$  are positively (negatively) quadrant dependent if

$$P(X_1 > x_1, X_2 > x_2) \geq (\leq) P(X_1 > x_1) P(X_2 > x_2) \quad (1.1.1)$$

or equivalently

$$P(X_1 \leq x_1, X_2 \leq x_2) \geq (\leq) P(X_1 \leq x_1) P(X_2 \leq x_2). \quad (1.1.2)$$

Stronger notions of bivariate positive or negative dependence were later developed by Esary, Proschan and Walkup (1967), and imply a strong form of dependence. This concept has been found to be very useful in obtaining the reliability of a system (see for example Barlow and Proschan (1975), Chapter 2, section 2 and 3).

Recently, Ahmed et al.(1978b) obtained multivariate versions of the bivariate positive versions of the bivariate positive dependence as described in the papers of Lehmann (1966), and Esary and Proschan (1972). They gave three definitions of positive dependence, each stronger than the preceding one.

Definition 1.1.1. The random variables  $X_1, \dots, X_n$  are said to be mutually positive orthant dependent if

$$P\left(\bigcap_{i=1}^n (X_i > x_i)\right) \geq \prod_{i=1}^n P(X_i > x_i) \quad (1.1.3)$$

for all real numbers  $x_1, \dots, x_n$ . Dykstra et al.(1973) have defined

$X_1, \dots, X_n$  ( $n \geq 2$ ) to be POD if

$$P\left(\bigcap_{i=1}^n (X_i \leq x_i)\right) \geq \prod_{i=1}^n P(X_i \leq x_i) \quad (1.1.4)$$

for all real numbers  $x_1, \dots, x_n$ . For  $n = 2$ , both (1.1.3) and (1.1.4) agree with Lehmann's definition of positive quadrant dependence.

In what follows we use the word 'increasing' for 'nondecreasing' and 'decreasing' for 'nonincreasing'.

Definition 1.1.2. A sequence  $X_1, \dots, X_n$  of random variables is said to be right tail increasing in sequence (RTIS) if for all real numbers  $x_i, i = 2, \dots, n$

$$P(X_i > x_i \mid X_1 > x_1, \dots, X_{i-1} > x_{i-1}) \quad (1.1.5)$$

is increasing in  $x_1, \dots, x_{i-1}$ . For  $n = 2$  this says  $X_2$  is right tail increasing (RTI) in  $X_1$  according to Esary and Proschan (1972).

Similarly the sequence  $X_1, \dots, X_n$  is said to be left tail decreasing in sequence (LTDS) if for all real numbers  $x_i, i = 2, \dots, n$

$$P(X_i \leq x_i \mid X_1 \leq x_1, \dots, X_{i-1} \leq x_{i-1}) \quad (1.1.6)$$

is decreasing in  $x_1, \dots, x_{i-1}$ . For  $n = 2$  this says  $X_2$  is left tail (LTD) in  $X_1$  according to Esary and Proschan (1972).

Definition 1.1.3. The random variables  $X_1, \dots, X_n$  are said to be conditionally increasing in sequence (CIS) if for  $i = 2, \dots, n$ , all real numbers  $x_i$ ,

$$P(X_i > x_i | X_1 = x_1, \dots, X_{i-1} = x_{i-1}) \quad (1.1.7)$$

is increasing in  $x_1, \dots, x_{i-1}$ . In the special case  $n = 2$ ,  $X_2$  is said to be positive regression dependent (PRD) in  $X_1$  in the terminology of Lehmann (1966).

Ahmed et al.(1978b) have shown that  $CIS \Rightarrow POD$  and  $RTIS \Rightarrow POD$ . Also they have demonstrated with the aid of examples that  $RTIS \not\Rightarrow CIS$  and vice versa. We introduce two other definitions useful in the context of positive dependence.

Definition 1.1.4. (Karlin 1968). A function  $f: R^2 \rightarrow [0, \infty]$  is totally positive of order 2 (TP<sub>2</sub>) if

$$\begin{vmatrix} f(x_1, y_1) & f(x_1, y_2) \\ f(x_2, y_1) & f(x_2, y_2) \end{vmatrix} \geq 0 \quad (1.1.8)$$

for each choice  $x_1 < x_2, y_1 < y_2$ .

Definition 1.1.5. (Esary, Proschan and Walkup (1967)). The random

variables  $X_1, \dots, X_n$  ( $n \geq 2$ ) are associated if

$$\text{Cov}(f(X_1, \dots, X_n), g(X_1, \dots, X_n)) \geq 0 \quad (1.1.9)$$

for all increasing real valued function  $f, g$  for which the covariance exist.

Ahmed et al.(1978b) have proved the following lemma which provides a chain of implications in some of these definitions of positive dependence.

Lemma 1.  $f_{X_1, \dots, X_n}(x_1, \dots, x_n)$  is  $TP_2$  in pairs  $\Rightarrow X_1, \dots, X_n$  are CIS  $\Rightarrow X_1, \dots, X_n$  are associated  $\Rightarrow X_1, \dots, X_n$  are POD.

In the bivariate case stronger results have been proved by Esary and Proschan (1972).

Result 1.  $TP_2 \Rightarrow PRD \Rightarrow LTD \Rightarrow \text{Associated} \Rightarrow PQD$ .

Result 2.  $TP_2 \Rightarrow PRD \Rightarrow RTI \Rightarrow \text{Associated} \Rightarrow PQD$ .

Next we give a few well-known multivariate distributions which exhibit some positive dependence among component variables.

Example 1. Let  $(X_1, \dots, X_n)$  have a multinormal distribution with mean vector  $(\mu_1, \dots, \mu_n)$  and variance-covariance matrix  $\Sigma$  which is positive definite. Let  $R = ((r_{ij})) = \Sigma^{-1}$ . Suppose  $r_{ij} \leq 0$  for all  $1 \leq i \leq j \leq n$ . Then the joint probability density function of  $(X_1, \dots, X_n)$  is  $TP_2$  in each pair of arguments for fixed values of the remaining arguments.

Example 2. Marshall-Olkin (1967) multivariate exponential distribution given by

$$P(X_1 > x_1, \dots, X_n > x_n) = \exp \left[ -\sum_{i=1}^n \lambda_i x_i - \sum_{1 \leq i < j \leq n} \lambda_{ij} \max(x_i, x_j) - \dots - \lambda_{12\dots n} \max(x_1, \dots, x_n) \right],$$

where  $\lambda_i \geq 0$  ( $1 \leq i \leq n$ ),  $\lambda_{ij} \geq 0$  ( $1 < i < j \leq n$ ),  $\dots$ ,  $\lambda_{12\dots n} \geq 0$ .

Example 3. A general family of positively dependent multivariate distributions is given by Farlie-Gumbel-Morgenstern system (see Johnson and Kotz 1975).

Consider  $n$  random variables  $X_1, \dots, X_n$  with joint probability density function of the form

$$f(x_1, \dots, x_n) = \prod_{j=1}^n f_j(x_j) \left[ 1 + \sum_{1 < j_1 < j_2 \leq n} a_{j_1, j_2} (1 - 2F_{j_1}(x_{j_1})) (1 - 2F_{j_2}(x_{j_2})) + \dots + a_{12\dots n} \prod_{j=1}^n (1 - 2F_j(x_j)) \right],$$

where the  $F_j$ 's are distribution functions with corresponding probability density functions  $F_j$  and constant  $a$ 's satisfying

$$\sum_{j_1 < j_2}^{r-1} a_{j_1, j_2} + \sum_{j_1 < j_2 < j_3}^{r-2} a_{j_1, j_2, j_3} + \dots + a_{a_1, \dots, a_r} \geq 0 \quad (1.1.10)$$

for any subset  $r$ ,  $(a_1, \dots, a_r)$  from  $(1, 2, \dots, n)$ . For positive dependence both

$$\sum_{j_1 < j_2}^{r-1} a_{j_1, j_2} - \sum_{j_1 < j_2 < j_3}^{r-2} a_{j_1, j_2, j_3} + \dots + (-1)^r a_{a_1, \dots, a_r} \geq 0 \quad (1.1.11)$$

for all subsets of  $r$  integers  $(a_1, \dots, a_r)$  from  $(1, 2, \dots, n)$  and  $r = 2, 3, \dots, n$  and (1.1.10) must hold.

Nonparametric classes of life distributions are playing a very important role in reliability analysis. In general, life length of a system of components is random, and this leads to the study of life distributions. The first effort to gain nonparametric classes of life distributions was due to Barlow et al. (1963). They classified life distributions according to the following definition:

Definition 1.1.6. The distribution function  $F$  or the survival function  $\bar{F} = 1 - F$  is said to be or to have an increasing failure rate (decreasing failure rate) if  $\frac{\bar{F}(x+t)}{\bar{F}(t)}$  is decreasing (increasing) in  $t$  whenever  $x > 0$ .

Later, Birnbaum et al. (1966) introduced two other nonparametric classes of life distributions according to the following definition:

Definition 1.1.7. The distribution function  $F$  is said to be increasing failure rate average (decreasing failure rate average) if  $[\bar{F}(t)]^{1/t}$  is decreasing (increasing) in  $t > 0$ .

Finally, Bryson and Siddiqui (1969) and Marshall and Proschan (1972) defined four other nonparametric classes of life distributions as the following:

Definition 1.1.8. The distribution function  $F$  is said to be new better than used (new worse than used) if

$$\overline{F}(x) \overline{F}(t) \geq \overline{F}(x+t) \quad ((\overline{F}(x) \overline{F}(t) \leq \overline{F}(x+t)) \text{ for all } x, t \geq 0.$$

Definition 1.1.9. The distribution function  $F$  is said to be new better than used in expectation (new worse than used in expectation) if

$$\infty > \int_0^{\infty} \overline{F}(x) dx \geq \int_0^{\infty} \frac{\overline{F}(t+x)}{\overline{F}(t)} dx \quad (\infty > \int_0^{\infty} \overline{F}(x) dx \leq \int_0^{\infty} \frac{\overline{F}(t+x)}{\overline{F}(t)} dx) \quad (1.1.12)$$

for all  $t > 0$ .

The following implications are readily checked:

Increasing failure rate (IFR)  $\Rightarrow$  Increasing failure rate average (IFRA)

$\Rightarrow$  New better than used (NBU)  $\Rightarrow$  New better than used in expectation (NBUE) and

Decreasing failure rate (DFR)  $\Rightarrow$  Decreasing failure rate average (DFRA)

$\Rightarrow$  New worse than used (NWU)  $\Rightarrow$  New worse than used in expectation (NWUE).

Next I present a few well-known life distributions which satisfy the above definitions.

Example 4. The Weibull distribution with survival function

$$F_a(t) = 1 - e^{-(\lambda t)^a} \text{ for } t \geq 0, \text{ where } \lambda, a > 0. \quad (1.1.13)$$

is IFR for  $a \geq 1$  and DFR for  $0 < a \leq 1$ . The special case  $a = 1$  leads to the exponential distribution, which is both IFR and DFR.

Example 5. The Gamma distribution with distribution function  $G_{\lambda, a}(t)$

$$G_{\lambda, a}(t) = \int_0^t \frac{\lambda^a x^{a-1}}{\Gamma(a)} e^{-\lambda x} dx \text{ where } \lambda, a > 0 \quad (1.1.14)$$

is IFR for  $a > 1$  and DFR for  $0 < a < 1$ .

Example 6. Assume a device is subject to shocks occurring in time according to a Poisson process with rate  $\lambda$ . Suppose the probability of surviving  $k$  shocks is  $\bar{P}_k$ , where  $1 = \bar{P}_0 \geq \bar{P}_1 \geq \dots$ . Then, the survival probability  $\bar{H}(t)$  of the device corresponding to the time interval  $[0, t]$  is given by

$$\bar{H}(t) = \sum_{k=0}^{\infty} \bar{P}_k \frac{e^{-\lambda t}}{k!} (\lambda t)^k \quad (1.1.15)$$

for  $t \geq 0$ . Now, let  $\bar{P}_k$  satisfy

$$\bar{P}_{k+\ell} \leq (\geq) \bar{P}_k \bar{P}_\ell \quad (1.1.16)$$

for  $k = 0, 1, 2, \dots$ ;  $\ell = 0, 1, 2, \dots$ .

Then  $H$  is NBU (NWU). Similarly if  $\bar{P}_k$  satisfy

$$\bar{P}_k \sum_{j=0}^{\infty} \bar{P}_j \geq (\leq) \sum_{j=k}^{\infty} \bar{P}_j \quad (1.1.17)$$

for  $k = 0, 1, 2, \dots$ . Then  $H$  is NBUE (NWUE).

For a proof of these claims, see Barlow and Proschan (1975, Chapter 6, pp. 160-161).

The assumption that the component life lengths are independent is often questionable at best, so it is important to understand multivariate extensions of the above classes of life distributions.



Extensions of the IFR and DFR from univariate to multivariate are due to Thompson and Brindley, 1972. Later, Marshall (1975) also generalized the concepts of IFR and DFR to the multivariate case in various ways. We list below some of the definitions which have a direct physical interpretation. In what follows, let  $\underline{x} = (x_1, \dots, x_n)$ ,  $\underline{t} = (t_1, \dots, t_n)$ ,  $\underline{1} = (1, \dots, 1)$  and  $\bar{F}_n(x_1, \dots, x_n) = P(X_1 > x_1, \dots, X_n > x_n)$ .

Definition 1.1.10. A multivariate distribution function  $F_n$  defined on the positive orthant is:

(1.1.10.1 multivariate increasing (decreasing) failure rate very strong,

MIFR-VS (MDFR-VS), if  $\frac{\bar{F}_n(\underline{x} + \underline{t})}{\bar{F}_n(\underline{t})}$  is decreasing (increasing) in  $\underline{t}$ ,  
for all  $\underline{t}, \underline{x} \geq \underline{0}$  and  $\bar{F}_n(\underline{t}) > 0$ ;

(1.1.10.2 multivariate increasing (decreasing) failure rate strong,

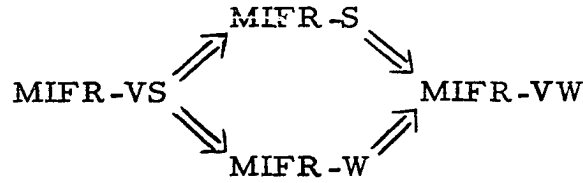
MIFR-S (MDFR-S), if  $\frac{\bar{F}_n(\underline{x} \underline{1} + \underline{t})}{\bar{F}_n(\underline{t})}$  is decreasing (increasing) in  $\underline{t}$ ,  
for all  $\underline{t} > \underline{0}$ ,  $\underline{x} > \underline{0}$  and  $\bar{F}_n(\underline{t}) > 0$ ;

(1.1.10.3 multivariate increasing (decreasing) failure rate

MIFR-W (MDFR-W), if  $\frac{\bar{F}_n(\underline{t} \underline{1} + \underline{x})}{\bar{F}_n(\underline{t} \underline{1})}$  is decreasing (increasing) in  $\underline{t}$ ,  
for all  $\underline{x} > \underline{0}$ ,  $\underline{t} > \underline{0}$ ,  $\bar{F}_n(\underline{t}) > 0$ , together with the same condition on all marginal survival functions;

(1.1.10.4 multivariate increasing (decreasing) failure rate very weak, MIFR-VW (MDFR-VW), if  $\frac{\bar{F}_n(x_1 + t_1)}{\bar{F}_n(t_1)}$  is decreasing (increasing) in  $t$ , for all  $t, x > 0$ ,  $\bar{F}_n(\cdot) > 0$ , together with the same condition on all marginal survival functions.

The following chain of implications between the above variations of the MIFR distribution can be verified. Analogous implications follow for the MDFR definitions



It is of interest to mention that if the survival times  $X_1, \dots, X_n$  are jointly IFR-VS, then

$$\bar{F}_n(x) \leq P(X_i > x_i)$$

is a kind of negative dependence.

## 1.2. Outline

Although there seems to be an abundance of literature on positive dependence, virtually very little seems to have been done on negative dependence. One reason might be that positive dependence among components is often built in many important multivariate distributions useful in describing various physical situations. Mention may be made of

the celebrated Marshall-Olkin (1967) multivariate exponential distribution. Also, positive dependence essentially goes hand in hand with an exchangeable sequence of random variables (see e.g. Shaked (1977)). Further, the usefulness of positive dependence in hypothesis testing, confidence estimation and reliability theory is well-known.

In spite of the importance of positive dependence in statistics and probability, this does not, by any means, exhaust all possible situations in statistical theory and practice. For example, a multinormal distribution with all negative pairwise correlation coefficients, cannot be expected to exhibit positive dependence among the component variables in any sense. The comment applies to other distributions like the multinomial and the Dirichlet. We shall also see that a broad subclass of the multidimensional Farlie-Gumbel-Morgenstern family of distributions (see Johnson and Kotz (1975)) exhibits negative dependence among the component variables in a very strong sense. We shall also see that negative dependence is useful in deriving certain reliability bounds.

We have introduced in Chapter 2 various notions of negative dependence and have studied their properties and interrelationships. In section 1, we have introduced the notions of negative orthant dependence (NOD) and strong NOD (SNOD). In Section 2, we have

defined the concepts of right tail decreasing in sequence (RTDS) and left tail increasing in sequence (LTIS), random variables, have studied their properties, and have discussed their inter-relationships with NOD and SNOD. In Section 3, we have introduced the concept of conditionally decreasing in sequence (CDS), random variables, and have studied its relationship with the previously introduced notions of negative dependence. The applications of these concepts to statistics and reliability are demonstrated in Section 4.

We have studied the ordering of negative quadrant dependence (NQD) in Chapter 3. The definition and some basic properties of NQD ordering are developed in Section 1. In conformity with one's intuition, it is shown in this section that the least NQD bivariate random vector corresponds to mutually independent random variables, while the most NQD bivariate random vector corresponds to the case in which one of the random variables is a nonincreasing function of the other.

Next, in Section 2, we have considered a family of bivariate distributions with specified marginals, the number of the family depending on a certain parameter, say  $\lambda$ . As  $\lambda$  increases, the corresponding distribution, say  $H_\lambda$  becomes increasingly NQD.

This concept is illustrated with the aid of certain examples which include a certain type of bivariate Farlie-Gumbel-Morgenstern distributions, certain types of bivariate distribution on pdf's constant on ellipses and others. Certain closure properties of the NQD ordering are derived in Section 3. It is shown that the NQD ordering is closed under convolution, mixture of a certain type, nondecreasing transformations of individual random variables, and limit in distribution. Finally several applications of the NQD ordering are given in Section 4.

The various notions of multivariate new better than used (MNBU) and multivariate new better than used in expectation (MNBUE) distribution have been considered in Chapter 4. In Section 1, we have first introduced various existing definitions of MNBU involving a certain hierarchy, and have described their physical implications. We have also examined in this section how several important classes of life distributions satisfy one or the other definition of the MNBU. Various closure properties of the MNBU distributions under the different definitions are studied in Section 2.

It is known in the univariate case that the class of IFRA distributions satisfies the NBU property. It is examined in Section 3 how far the multivariate IFRA distribution introduced by Esary

and Marshall (1979) leads to one or the other MNBU definition as introduced in Section 1. In Section 4 we have examined how some of the shock models with shocks governed by a general counting process satisfying certain conditions leads to a MNBU survival function. The corresponding result of Marshall and Shaked (1979) turns out to be a special case of one of our results.

In Section 5, we have taken three definitions of the multivariate NBUE (MNBUE) distributions from Buchanan and Singpurwalla (1977) and have given physical meanings to these definitions. We have also demonstrated with the aid of an example that contrary to what is claimed by Buchanan and Singpurwalla (1977), certain chains of implications in these definitions of MNBUE fails to hold in general. It is known in the univariate case that NBU implies NBUE. The relationship among the various definitions of the MNBU and MNBUE is also discussed in this section. Also, certain important classes of life distributions satisfying one or the other MNBUE definitions are given in Section 6; various closure properties of the MNBUE distribution under the different definitions is studied. Finally, in Section 7, we have introduced certain shock models leading to MNBUE survival function.

The respective duals of the NBU and NBUE distributions are

the NWU and NWUE distributions. In Section 1 and 5 multivariate NWU (MNWU) and multivariate NWUE (MNWUE) definitions are given parallel to the MNBUE and MNBUE definitions. We have discussed the closure properties of the MNWU and MNWUE distributions whenever appropriate. Generating MNWU and MNWUE distributions through shock models is also discussed.

## 2. MULTIVARIATE NEGATIVE DEPENDENCE

### 2.1. Negative Orthant Dependence (NOD) and Strong Negative Orthant Dependence

We start with the definition of NOD.

Definition 2.1.1. The random variables  $X_1, \dots, X_n$  ( $n \geq 2$ ) are said to be mutually negative orthant dependent (NOD) if

$$P(X_1 > x_1, \dots, X_n > x_n) \leq \prod_{i=1}^n P(X_i > x_i) \quad (2.1.1)$$

for all real numbers  $x_1, \dots, x_n$ . For  $n = 2$ , NOD agrees with negative quadrant dependence (NQD) as defined in Lehmann (1966).

The definition (2.1.1) is motivated from its POD counterpart as defined in (1.1.1). Alternately, motivated from (1.2.1) one could define NOD as

$$P\left(\bigcap_{i=1}^n (X_i \leq x_i)\right) \leq \prod_{i=1}^n P(X_i \leq x_i) \quad (2.1.2)$$

for all real numbers  $x_1, \dots, x_n$ .

For  $n = 2$ , the definitions given in (2.1.1) and (2.1.2) are equivalent. However, as one might expect, these definitions do not in general agree for  $n \geq 3$ . The following example illustrates this.

Let  $X_1$ ,  $X_2$  and  $X_3$  be three random variables assuming the values  $(0, 1, 1)$ ,  $(1, 0, 1)$ ,  $(1, 1, 0)$  and  $(0, 0, 0)$  each with probability  $\frac{1}{4}$ . Then,  $P(X_1 > 0, X_2 > 0, X_3 > 0) = 0 < \frac{1}{8} = P(X_1 > 0) P(X_2 > 0) P(X_3 > 0)$ ;



but  $P(X_1 \leq 0, X_2 \leq 0, X_3 \leq 0) = \frac{1}{4} > \frac{1}{8} = P(X_1 \leq 0) P(X_2 \leq 0) P(X_3 \leq 0)$ .

There are, however, random variables for which both (2.1.1) and (2.1.2) are satisfied. We shall see examples of this later. Such random variables definitely exhibit a stronger form of negative dependence than the usual NOD and this motivates us to introduce the following definition.

Definition 2.1.2. The random variables  $X_1, \dots, X_n$  ( $n \geq 2$ ) are said to be strong negative orthant dependent (SNOD) if both (2.1.1) and (2.1.2) hold. We shall see later how SNOD is useful in deriving reliability bounds.

We list below a number of important properties of NOD and SNOD variables.

- (N1) Any set of independent random variables is SNOD.
- (N2) Any subset of NOD (SNOD) random variables of size  $\geq 2$  is NOD (SNOD).
- (N3) If  $X_1, \dots, X_n$  are NOD (SNOD) and  $g_1, \dots, g_n$  are real valued increasing functions, then  $g_1(X_1), \dots, g_n(X_n)$  are NOD (SNOD).
- (N4) The union of independent sets of NOD (SNOD) random variables is NOD (SNOD).

Remark 1. A set consisting of a single random variable is not NOD.

It might be of interest to note that pairwise NOD does not imply mutual

NOD. As an example, consider the sample space  $\Omega$  to be the set of equally likely integers  $\omega$  with  $1 \leq \omega \leq 8$ . Let  $A_1 = \{1, 2, 3, 4, 5\}$ ,  $A_2 = \{3, 4, 5, 6, 7\}$  and  $A_3 = \{2, 4, 5, 6, 8\}$ . Let  $X_i$  denote the indicator function of the set  $A_i$  ( $i = 1, 2, 3$ ). Then,  $P(X_i > 0, X_j > 0) = \frac{3}{8} < \frac{25}{64} = P(X_i > 0) P(X_j > 0)$ ,  $1 \leq i \neq j \leq 3$ . But  $P(X_1 > 0, X_2 > 0, X_3 > 0) = \frac{2}{8} > (\frac{5}{8})^3 = P(X_1 > 0) P(X_2 > 0) P(X_3 > 0)$ . (2.1.3)

Before proving any further properties of NOD random variables, we need the following definition.

Definition 2.1.3. A random vector  $\underline{Y}$  is stochastically increasing (decreasing) in the random variable  $\underline{X}$  if  $E(f(\underline{Y}) | \underline{X} = \underline{x})$  is increasing (decreasing) in  $\underline{x}$  for all real valued increasing functions  $f$ . We shall use the abbreviation SI and SD for stochastically increasing and decreasing respectively.

The following theorem gives a sufficient condition for NQD ness.

Theorem 1. Let (a)  $(X_1, X_2)$  given  $\lambda$ , a scalar random variable, be conditionally NQD, and (b)  $X_1$  be SI in  $\lambda$  and  $X_2$  be SD in  $\lambda$  or (b)'  $X_1$  be SD in  $\lambda$  and  $X_2$  be SI in  $\lambda$ . Then,  $(X_1, X_2)$  is NQD.

Proof.

$$\begin{aligned} \text{COV}[f(X_1), g(X_2)] &= \text{COV}_{\lambda}[E\{f(X_1)|\lambda\}, E\{g(X_2)|\lambda\}] \\ &\quad + E_{\lambda}[\text{COV}(f(X_1), g(X_2))|\lambda] \end{aligned} \quad (2.1.4)$$

The first term in the right hand side of (2.1.4) is nonpositive from (b) or (b)' for increasing  $f$  and  $g$ . For such  $f$  and  $g$ , the second term is nonpositive using assumption (a). Noting that  $X_1$  and  $X_2$  are NQD if and only if  $\text{COV}(f(X_1), g(X_2)) \leq 0$  for all increasing  $f$  and  $g$  (See Lehmann (1966), Theorem 1 (i)), the result follows.

Corollary. Let  $(X_1, X_2)$  be NQD, and let  $Z$  be independent of  $(X_1, X_2)$ . Define  $X = X_1 + aZ$ ,  $Y = X_2 + bZ$ . Then  $ab \leq 0 \Rightarrow (X, Y)$  is NQD.

Proof. Let  $a \geq 0$ ,  $b \leq 0$ . Then  $X_1 + aZ$  is SI in  $Z$  and  $X_2 + bZ$  is SD in  $Z$ . Since  $(X, Y)$  given  $Z$  is NQD, by Theorem 1,  $(X, Y)$  is NQD. Similarly, one handles the case  $a \leq 0$ ,  $b \geq 0$ .

Remark 2. The following example shows that the converse of the above corollary does not hold. Suppose  $(X_1, X_2)$  has a bivariate normal distribution with zero means, unit variances and correlation coefficient  $-\frac{3}{4}$ . Hence  $(X_1, X_2)$  is NQD. Let  $Z$  be  $N(0, \frac{1}{4})$  variable distributed independently of  $(X_1, X_2)$ . Let  $X = X_1 + Z$ ,  $Y = X_2 + Z$ . Then  $\text{COV}(X, Y) = -\frac{1}{2} < 0$ . So,  $(X, Y)$  is NQD.

The following theorem shows that for binary random variables, NQD ness is equivalent to nonpositiveness of the correlation coefficient between the two random variables.

Theorem 2. Let  $X_1$  and  $X_2$  be two binary random variables. Then  $(X_1, X_2)$  is NQD if and only if  $\text{COV}(X_1, X_2) \leq 0$ .

Proof. Without loss of generality, assume that each  $X_1$  and  $X_2$

assumes the values 0 and 1. Then  $\text{COV}(X_1, X_2) \leq 0 \Leftrightarrow P(X_1 = 1, X_2 = 1) \leq P(X_1 = 1) P(X_2 = 1)$  i. e.  $P(X_1 > 0, X_2 > 0) \leq P(X_1 > 0) P(X_2 > 0)$ .

Since,  $P(X_i > 0, X_j > 1) = P(X_i > 0) P(X_j > 1)$   $1 \leq i \neq j \leq 2$ , and  $P(X_1 > 1, X_2 > 1) = 0 = P(X_1 > 1) P(X_2 > 1)$ , the result follows.

Notation. Let  $\underline{\tilde{x}} = (x_1, \dots, x_n)$ ,  $\underline{\tilde{y}} = (y_1, \dots, y_n)$ . We say  $\underline{\tilde{x}} \geq \underline{\tilde{y}}$  if  $x_\ell \geq y_\ell$  for all  $\ell = 1, \dots, n$ .

For proving the next theorem, we need the following definition.

Definition 2.1.4. A random vector  $\underline{\tilde{Y}}$  is said to be stochastically right tail increasing (decreasing) in the random vector  $\underline{\tilde{X}}$  if  $E[f(\underline{\tilde{Y}}) | \underline{\tilde{X}} > \underline{\tilde{x}}]$  is increasing (decreasing) in  $\underline{\tilde{x}}$  for every real valued increasing  $f$ .

Theorem 3. Let (a)  $\underline{\tilde{X}}$  be NOD, (b)  $Y_1, \dots, Y_m$  be conditionally independent given  $\underline{\tilde{X}}$ , and (c)  $Y_\ell$  be stochastically right tail decreasing in  $\underline{\tilde{X}}$  for all  $\ell = 1, \dots, m$ . Then, (i)  $(\underline{\tilde{X}}, \underline{\tilde{Y}})$  is NOD; (ii)  $\underline{\tilde{Y}}$  is NOD.

Proof. 
$$P\left(\bigcap_{j=1}^n (X_j > x_j), \bigcap_{\ell=1}^m (Y_\ell > y_\ell)\right) = P\left(\bigcap_{\ell=1}^m (Y_\ell > y_\ell) \mid \bigcap_{j=1}^n (X_j > x_j)\right) P\left(\bigcap_{j=1}^n (X_j > x_j)\right)$$
$$= \prod_{\ell=1}^m P(Y_\ell > y_\ell \mid \bigcap_{j=1}^n (X_j > x_j)) P\left(\bigcap_{j=1}^n (X_j > x_j)\right), \text{ using (b)}$$
$$\leq \prod_{\ell=1}^m P(Y_\ell > y_\ell) \prod_{j=1}^n P(X_j > x_j), \text{ using (c) and (a).}$$

(ii) It follows from (i) by making  $x_j \rightarrow -\infty$ ,  $j = 1, \dots, n$ .

The next theorem demonstrates the preservation of the NOD

property under limits.

Theorem 4. Let  $\{\tilde{X}_n : n \geq 1\}$  be a sequence of NOD  $p$ -dimensional random vectors with distribution functions  $H_n$  such that  $H_n \rightarrow H$  weakly as  $n \rightarrow \infty$  where  $H$  is the distribution function of a random vector  $\tilde{X} = (X_1, \dots, X_p)$ . Then,  $\tilde{X}$  is NOD.

Proof. For any real  $x_1, \dots, x_p$ , writing  $\tilde{X}_n = (X_{1n}, \dots, X_{pn})$ ,  $n \geq 1$ ,

$$\begin{aligned} P(X_1 > x_1, \dots, X_p > x_p) &= \lim_{n \rightarrow \infty} P(X_{1n} > x_1, \dots, X_{pn} > x_p) \\ &\leq \lim_{n \rightarrow \infty} \prod_{j=1}^p P(X_{jn} > x_j) = \prod_{j=1}^p P(X_j > x_j). \end{aligned}$$

In what follows if  $H$  is the distribution function for a  $p$ -dimensional random vector  $\tilde{X} = (X_1, \dots, X_p)$ , we write

$$\bar{H}(x_1, \dots, x_p) = P(X_1 > x_1, \dots, X_p > x_p).$$

Theorem 5. Let  $H_0$  and  $H_1$  be two multivariate NOD distributions both having the same one-dimensional marginals. Then if  $\bar{H}_a = a \bar{H}_0 + (1-a) \bar{H}_1$ ,  $a \in (0, 1)$ ,  $H_a$  is also NOD.

Proof. By definition, the one-dimensional marginals of  $\bar{H}_a$  are the same as those of  $\bar{H}_0$  or  $\bar{H}_1$ . Also

$$\begin{aligned} P_{\bar{H}_a}(X_1 > x_1, \dots, X_p > x_p) &= a P_{\bar{H}_0}(X_1 > x_1, \dots, X_p > x_p) \\ &\quad + (1-a) P_{\bar{H}_1}(X_1 > x_1, \dots, X_p > x_p) \\ &\leq a \prod_{j=1}^p P_{\bar{H}_a}(X_j > x_j) + (1-a) \prod_{j=1}^p P_{\bar{H}_a}(X_j > x_j) \end{aligned}$$

$$= \prod_{j=1}^P P_{H_a}^-(X_j > x_j).$$

Hence,  $H_a$  is NOD.

## 2.2. Right Tail Decreasing in Sequence (RTDS) and Left Tail Increasing in Sequence (LTIS)

In this section we present some other notions of negative dependence stronger than the NOD.

Definition 2.2.1. A sequence  $\{X_1, \dots, X_n\}$  of random variables is said to be right tail decreasing in sequence (RTDS) if for all real  $x_i$ ,  $i = 1, \dots, n-1$ ,

$$P(X_{i+1} > x_{i+1} \mid \bigcap_{j=1}^i (X_j > x_j)) \quad (2.2.1)$$

is decreasing in  $x_1, \dots, x_i$ . If (2.2.1) holds for  $n = 2$ ,  $X_2$  is said to be right tail decreasing (RTD) in  $X_1$ . We now show that RTDS  $\Rightarrow$  NOD.

Theorem 6. If  $(X_1, \dots, X_n)$  ( $n \geq 2$ ) is RTDS, then it is also NOD.

Proof. 
$$P\left(\bigcap_{j=1}^n (X_j > x_j)\right) = P(X_1 > x_1) \prod_{i=2}^n P(X_i > x_i \mid \bigcap_{j=1}^{i-1} (X_j > x_j))$$

$$\leq \prod_{i=1}^n P(X_i > x_i)$$

making  $x_j \rightarrow -\infty$  ( $j = 1, \dots, i-1$ ). The result follows.

Parallel to the RTDS, we now define the left tail increasing sequence (LTIS) property.

Definition 2.2.2.  $(X_1, \dots, X_n)$  is said to be LTIS if for all real  $x_i$ ,  $i = 1, \dots, n-1$ ,

$$P(X_{i+1} \leq x_{i+1} | \bigcap_{j=1}^i (X_j \leq x_j)) \quad (2.2.2)$$

is increasing in  $x_1, \dots, x_i$ . If (2.2.2) holds for  $n = 2$ , the property is referred to as the LTI property. Note that LTIS implies (2.1.2.).

The next example shows that  $RTD \not\Rightarrow LTI$ . Let  $(X, Y)$  be two random variables having a joint probability function as follows.

$\begin{smallmatrix} X \\ Y \end{smallmatrix}$	0	1	2	3	
0	.15	.10	.20	.25	.70
1	.10	.10	.10	0	.30
	.25	.2	.3	.25	1

Then,  $P(Y > 0 | X \geq 0) = \frac{3}{10}$ ,  $P(Y > 0 | X \geq 1) = \frac{44}{165}$ ,

$P(Y > 0 | X \geq 2) = \frac{30}{165}$ ,  $P(Y > 0 | X \geq 3) = 0$ . Hence,  $Y$  is RTD in  $X$ .

However,  $P(Y = 0 | X \leq 0) = \frac{3}{5} > \frac{5}{9} = P(Y = 0 | X \leq 1)$ , hence,  $Y$  is not LTI in  $X$ .

The next example illustrates that  $LTI \not\Rightarrow RTD$ . Suppose  $(X, Y)$  has the joint probability function as follows.

$\begin{smallmatrix} X \\ Y \end{smallmatrix}$	0	1	2	3	
0	.10	.15	.2	.15	.60
1	.15	.10	.05	.10	.40
	.25	.25	.25	.25	1

(2.2.3)

Now  $P(Y = 0 | X \leq 0) = \frac{2}{5}$ ,  $P(Y = 0 | X \leq 1) = \frac{1}{2}$ ,  $P(Y = 0 | X \leq 2) = \frac{3}{5}$  and

$P(Y = 0 | X \leq 3) = \frac{3}{5}$ . Hence,  $Y$  is LTI in  $X$ . But

$P(Y = 1 | X > 1) = .3 < .40 = P(Y = 1 | X > 2)$ . Hence,  $Y$  is not RTD in  $X$ .

It is easy to show that  $LTI \Rightarrow NQD$ . However, for  $n \geq 3$ ,  $LTIS \not\Rightarrow NOD$ . The following example illustrates this.

Let  $X, Y$  and  $Z$  be three random variables such that conditional on  $X \leq x$  and  $Y \leq y$ ,  $Z$  has distribution function

$$P(Z \leq z | X \leq x, Y \leq y) = 1 - \exp\{-z(x+y)\}, \quad Z > 0, \quad (2.2.4)$$

while  $X$  and  $Y$  have the joint probability function as given in (2.2.3).

Since the right hand side of (2.2.4) is increasing in  $x$  and  $y$ , and from the previous example,  $(X, Y)$  is LTI,  $(X, Y, Z)$  taken in this sequence is LTIS. However, since  $P(Z \leq z | X \leq x) = 1 - e^{-z(x+1)}$  and  $P(Z \leq z | Y \leq y) = 1 - e^{-z(y+3)}$ , it follows after some algebra that

$$P(X > 2, Y > 0, Z > 2) = \exp(-8) - 1.35 \exp(-6) + .45 \exp(-4)$$

$$> (.1) \exp(-8) = P(X > 2) P(Y > 0) P(Z > 2).$$

We list below the following properties of RTDS.

- (R1) Any set of independent random variables is RTDS.
- (R2) Any subset of RTDS random variables is RTDS.
- (R3) If  $X_1, \dots, X_n$  is RTDS and  $g_1, \dots, g_n$  are increasing Borel measurable functions, then  $g_1(X_1), \dots, g_n(X_n)$  is RTDS.

The following theorem provides a characterization of RTD in the



bivariate case.

Theorem 7. Let  $X = (X_1, X_2)$ . Then  $X_2$  is RTD in  $X_1$

$\Leftrightarrow P(X_2 \geq x_2 | X_1 > x_1) \downarrow$  in  $x_1$  for all real  $x_2 \Leftrightarrow E[f(X_2) | X_1 > x_1]$   
 $\downarrow$  in  $x_1$  for all real valued increasing bounded continuous  $f$ .

Proof.  $X_2$  RTD in  $X_1 \Leftrightarrow P(X_2 > x_2 | X_1 > x_1) \downarrow$  in  $x_1$  for all real  $x_2$   
 $\Leftrightarrow P(X_2 \geq x_2 | X_1 > x_1) \downarrow$  in  $x_1$  for all real  $x_2$ .

Now assume  $E[f(X_2) | X_1 > x_1] \downarrow$  in  $x_1$  for all real valued increasing bounded continuous  $f$ . Define  $u_{x_2}^{(k)}(y) = 0, k[y - (x_2 - k^{-1})]$  or 1

according as  $y < x_2 - k^{-1}$ ,  $y \in [x_2 - k^{-1}, x_2]$  or  $y \geq x_2$ . Note that for each  $k$  and  $x_2$ ,  $u_{x_2}^{(k)}(y)$  is  $\uparrow$  in  $y$ . Hence,  $E[u_{x_2}^{(k)}(X_2) | X_1 > x_1] \downarrow$  in  $x_1$  for each fixed  $x_2$  and  $k$ . Also,  $u_{x_2}^{(k)}(y)$  is  $\downarrow$  in  $k$  for each fixed  $x_2$  and  $y$ , is bounded above by 1 and  $u_{x_2}^{(k)}(y) \downarrow I_{[y \geq x_2]}$  as  $k \rightarrow \infty$ ,

$I$  being the usual indicator function.

Hence applying the monotone convergence theorem for conditional expectation,

$$E \left[ I_{[X_2 \geq x_2]} | X_1 > x_1 \right] \text{ is } \downarrow \text{ in } x_1 \text{ i.e. } P(X_2 \geq x_2 | X_1 > x_1) \\ \text{is } \downarrow \text{ in } x_1.$$

Conversely if  $E \left[ I_{[X_2 \geq x_2]} | X_1 > x_1 \right]$  is  $\downarrow$  in  $x_1$  for each fixed  $x_2$ ,

it follows in succession that  $E(f(X_2) | X_1 > x_1)$  is  $\downarrow$  in  $x_1$  for every nonnegative simple function  $f$ , for every nonnegative increasing  $f$  and finally for every increasing  $f$  implying the result for every

bounded continuous increasing  $f$ .

Remark 3. We could have stated Theorem 7 with "all real valued increasing  $f$ " replacing "all real valued increasing bounded continuous  $f$ ". The proof of the theorem would then be simpler. But in later application we need the fact that " $E[f(X_2)|X_1 > x_1] \downarrow$  in  $x_1$  for all real valued bounded continuous  $f \Rightarrow P(X_2 \geq x_2 | X_1 > x_1) \downarrow$  in  $x_1$  for all real  $x_2$ " which would have then required a separate proof.

Remark 4. Theorem 7 can be generalized to a multivariate RTDS sequence of random variables.

Remark 5. A theorem similar to Theorem 7 can be proved for LTI or in general LTIS sequence of random variables.

The following theorem exhibits a RTD (or LTI) preservation property.

Theorem 8. Let (a)  $X_2$  be RTD (LTI) in  $X_1$  conditional on  $\lambda_{\sim}$ . Then  $X_2$  is RTD (LTI) in  $X_1$ .

Proof. Let  $x_1 < x'_1$ . Note that if the stated RTD property holds, then

$$\begin{aligned} P(X_2 > x_2 | X_1 > x_1) &= E_{\lambda_{\sim}} [P(X_2 > x_2 | X_1 > x_1, \lambda)] \\ &\geq E_{\lambda_{\sim}} [P(X_2 > x_2 | X_1 > x'_1, \lambda)] \\ &= P(X_2 > x_2 | X_1 > x'_1) \end{aligned}$$

for all  $x_2$ . Similarly, one handles the LTI case.

Corollary. Let  $X_2$  be RTD in  $X_1$ , and let  $Z$  be independent of  $(X_1, X_2)$ . Define  $X = X_1 + aZ$ ,  $Y = X_2 + bZ$ , where  $a, b$  are constants. Then  $Y$  is RTD in  $X$ .

Proof. Must show  $P(Y > y | X > x) \downarrow$  in  $x$  for all real  $y$ . But for  $x' < x''$ ,

$$\begin{aligned} P(Y > y | X > x') &= P(X_2 + bZ > y | X_1 + aZ > x') \\ &= E_Z P(X_2 > y - bZ | X_1 > x' - aZ, Z) \\ &\geq E_Z P(X_2 > y - bZ | X_1 > x'' - aZ, Z) \\ &= P(X_2 > y - bZ | X_1 + aZ > x''). \end{aligned}$$

This proves the result.

The next theorem shows that RTDS property is preserved under limits.

Theorem 9. Suppose  $H_n$  converges to  $H$ , and the sequence of multivariate random variables related to  $H_n$  is RTDS. Then the multivariate random variable related to  $H$  is RTDS.

Proof. Use Theorem 7 and the Helly-Bray Theorem.

Next note that if  $H_1$  and  $H_2$  are two multivariate RTDS distributions defined on  $R^p$  such that all the marginals  $< p$  for  $H_1$  and  $H_2$  are the same, then for any  $\alpha \in (0, 1)$ ,  $\bar{H}_\alpha(x_1, \dots, x_p)$   
 $= \alpha \bar{H}_1(x_1, \dots, x_p) + (1-\alpha) \bar{H}_2(x_1, \dots, x_p)$  is also RTDS. In the special case when  $H_1$  and  $H_2$  are two bivariate distributions with the

same marginals,  $\overline{H}_\alpha$  is RTD.

The final two theorems in this section relate to RTD and RTDS preservation properties.

Theorem 10. Let  $(U, V)$  be RTD,  $Z$  be independent of  $(U, V)$ , and let  $f$  and  $g$  each be a Borel measurable map from  $R^2$  to  $R$  with  $f(u, \cdot)$  increasing in  $u$  and  $g(\cdot, v)$  increasing in  $v$ . Define  $X = f(U, Z)$ ,  $Y = g(V, Z)$ . Then  $(X, Y)$  is RTD.

Proof.  $P\{Y > y | X > x\} = P\{g(V, Z) > y | f(U, Z) > x\}$ . Now,

$$P\{g(V, z) > y | f(U, z) > x, z\} \downarrow \text{ in } x.$$

Hence, using Theorem 8, the result follows.

Theorem 11. Let (a)  $\underset{\sim}{X} = (X_1, \dots, X_n)$  be a RTDS sequence of random variables,  $g_\ell: R \rightarrow R$  be a Borel measurable increasing function for each  $\ell = 1, \dots, n$ . (b)  $Z = (Z_1, \dots, Z_n)$  be a RTDS sequence of random variables which is independent of  $\underset{\sim}{X}$ . Define  $Y_\ell = g_\ell(X_\ell) + Z_\ell$ ,  $\ell = 1, \dots, n$ . Then  $Y_1, \dots, Y_n$  is RTDS.

Proof. First note that for  $\ell = 2, \dots, n$ ,

$$\begin{aligned} P\{Y_\ell > y_\ell | \bigcap_{j=1}^{\ell-1} (Y_j > y_j)\} &= E_Z \{P(Y_\ell > y_\ell | \bigcap_{j=1}^{\ell-1} (Y_j > y_j), \underset{\sim}{Z})\} \\ &= E_Z [P\{g_\ell(X_\ell) > y_\ell - Z_\ell | \bigcap_{j=1}^{\ell-1} (Y_j > y_j), \underset{\sim}{Z}\}]. \end{aligned}$$

Now by property R3,  $(g_1(X_1), \dots, g_n(X_n))$  is RTDS. Also,  $\underset{\sim}{Z}$  is RTDS.

Hence,  $E \left[ P \left\{ g_{\ell}(X_{\ell}) > y_{\ell} - Z_{\ell} \mid \bigcap_{j=1}^{\ell-1} (g_j(X_j) > y_j - Z_j), Z \right\} \right]$  is  $\downarrow$  in  $y_1, \dots, y_{\ell-1}$ .

### 2.3. Conditionally Decreasing in Sequence (CDS)

We introduce now another notion of negative dependence.

Definition 2.3.1. The random variables  $X_1, \dots, X_n$  are said to be conditionally decreasing in sequence (CDS) if for  $i = 2, 3, \dots, n$  and all real numbers  $x_i$ ,

$$P(X_i > x_i \mid X_1 = x_1, \dots, X_{i-1} = x_{i-1}) \quad (2.3.1)$$

is decreasing in  $x_1, \dots, x_{i-1}$ . In the case  $n = 2$ ,  $X_2$  is said to be negative regression dependent (NRD) in  $X_1$  in the terminology of Lehmann (1966).

The following lemma relates the PRD and the NRD properties of random variables.

Lemma 1. (i)  $(X_1, X_2)$  is NRD if and only if  $(-X_1, X_2)$  is PRD.

(ii)  $(X_1, X_2)$  is NRD if and only if  $(X_1, -X_2)$  is PRD.

Proof. (i) Let  $(X_1, X_2)$  be NRD. Let  $y_1 < y_2$ . Then

$$\begin{aligned} P(X_2 > x_2 \mid -X_1 = y_1) &= P(X_2 > x_2 \mid X_1 = -y_1) \\ &\leq P(X_2 > x_2 \mid X_1 = -y_2) \\ &= P(X_2 > x_2 \mid -X_1 = y_2). \end{aligned}$$

Hence  $(-X_1, X_2)$  is PRD. The other side is proved similarly.

(ii) The proof of (ii) is similar to the proof of (i).

Theorem 12. If  $(X_1, X_2)$  is NRD, then it is also RTD.

Proof.  $(X_1, X_2)$  is NRD  $\Leftrightarrow (X_1, -X_2)$  is PRD  $\Rightarrow (X_1, -X_2)$  is RTI. Also, it can be shown in the same way as the proof of Lemma 1 that  $(X_1, -X_2)$  is RTI  $\Leftrightarrow (X_1, X_2)$  is RTD.

Remark 6. One can similarly show that if  $(X_1, X_2)$  is NRD, then it is also LTI.

It is tempting to conjecture that  $CDS \Rightarrow RTDS$ . The following example shows that this is not always true.

Let  $X_1, X_2, X_3$  be three random variables such that conditional on  $X_1 = x_1$  and  $X_2 = x_2$ ,  $X_3$  has probability density function.

$$f(x_3) = x_2 \exp(-x_2 x_3), \quad x_3 > 0,$$

while  $X_1$  and  $X_2$  have the joint probability function as follows:

$X_2 \backslash X_1$	0	1	2	
1	0	.2	.3	.5
2	.2	0	.1	.3
3	.1	.1	0	.2
	.3	.3	.4	1

Then,  $P(X_2 > 1 | X_1 = 0) = 1$ ,  $P(X_2 > 1 | X_1 = 1) = \frac{1}{3}$ ,  $P(X_2 > 1 | X_1 = 2) = \frac{1}{4}$ ,

$P(X_2 > 2 | X_1 = 0) = \frac{1}{3}$ ,  $P(X_2 > 2 | X_1 = 1) = \frac{1}{3}$ ,  $P(X_2 > 2 | X_1 = 2) = 0$ . Hence,

$X_2$  is NRD in  $X_1$ . Also,  $P(X_3 > x | X_1 = x_1, X_2 = x_2) = \exp(-x_2 x) \downarrow$  in  $x_2$ .

Hence,  $(X_1, X_2, X_3)$  is CDS. However, using the identity

$$P(C|A \cup B) = P(C|A) \frac{P(A)}{P(A)+P(B)} + P(C|B) \frac{P(B)}{P(A)+P(B)}$$

when  $P(A) > 0$ ,  $P(B) > 0$ , and  $A$  and  $B$  are disjoint, one gets

$$\begin{aligned} P(X_3 > x | X_1 > 0, X_2 > 1) &= \frac{1}{2} [P(X_3 > x | X_1 = 1, X_2 = 3) \\ &\quad + P(X_3 > x | X_1 = 2, X_2 = 2)] \\ &= \frac{1}{2} [\exp(-3x) + \exp(-2x)], \end{aligned}$$

while

$$P(X_3 > x | X_1 > 1, X_2 > 1) = P(X_3 > x | X_1 = 2, X_2 = 2) = \exp(-2x)$$

so that

$$P(X_3 > x | X_1 > 1, X_2 > 1) > P(X_3 > x | X_1 > 0, X_2 > 1) \text{ for all } x > 0.$$

Hence,  $(X_1, X_2, X_3)$  is not RTDS.

As we mentioned in chapter one, the concept of  $TP_2$  defined in (1.1.4) has been found to be very useful in connection with positive dependence. For proving NRD it is convenient to verify a property similar to (1.1.4) with the inequality going in the opposite direction.

Theorem 13. Let  $(X, Y)$  have the joint probability density function

$f(x, y)$  which satisfies

$$\begin{vmatrix} f(x_1, y_1) & f(x_1, y_2) \\ f(x_2, y_1) & f(x_2, y_2) \end{vmatrix} \leq 0, \quad (2.3.2)$$

for every choice  $x_1 < x_2$ ,  $y_1 < y_2$ . Then  $(X, Y)$  is NRD.

Remark 7. An example of a bivariate density satisfying (2.3.2) is

$f(x, y) = x + y$ ,  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ , because the,  $f(x_1, y_1) f(x_2, y_2) - f(x_1, y_2) f(x_2, y_1) = -(x_2 - x_1)(y_2 - y_1) < 0$  for  $0 \leq x_1 < x_2 \leq 1$ ,  $0 \leq y_1 < y_2 \leq 1$ .

Proof of Theorem 13. For  $x_1 < x_2$ ,  $y_1 < y_2$ , we have,

$$\begin{aligned}
 & \begin{vmatrix} f(x_1, y_1) & f(x_1, y_2) \\ f(x_2, y_1) & f(x_2, y_2) \end{vmatrix} \leq 0 \\
 & \Rightarrow \begin{vmatrix} \int_t^\infty f(x_1, y_2) dy_2 & \int_t^\infty f(x_2, y_2) dy_2 \\ \int_{-\infty}^t f(x_1, y_1) dy_1 & \int_{-\infty}^t f(x_2, y_1) dy_1 \end{vmatrix} \geq 0 \\
 & \Leftrightarrow \begin{vmatrix} \int_t^\infty f(x_1, y) dy & \int_t^\infty f(x_2, y) dy \\ \int_{-\infty}^t f(x_1, y) dy & \int_{-\infty}^t f(x_2, y) dy \end{vmatrix} \geq 0 \\
 & \Leftrightarrow \begin{vmatrix} \int_t^\infty f(x_1, y) dy & \int_t^\infty f(x_2, y) dy \\ f_1(x_1) & f_1(x_2) \end{vmatrix} \geq 0
 \end{aligned}$$

( $f_1(x)$  denoting the marginal probability density function of  $X$ )



$$\Leftrightarrow \int_t^\infty f(x_1, y) dy / f_1(x_1) \geq \frac{\int_t^\infty f(x_2, y) dy}{f_1(x_2)}$$

$$\Leftrightarrow P(Y > t | X = x_1) \geq P(Y > t | X = x_2)$$

which shows that  $Y$  is NRD in  $X$ . By symmetry of the condition (2.3.2),  $X$  is NRD in  $Y$ .

In the definition of Karlin (1968), a function  $f: R^n \rightarrow [0, \infty]$  is said to be  $TP_2$  in pairs if for any pair  $(x_i, x_j)$ ,  $f(x_1, \dots, x_i, \dots, x_j, \dots, x_n)$  viewed as a function of  $(x_i, x_j)$  with the other arguments held fixed satisfies

$$\begin{vmatrix} f(x_1, \dots, x_i, \dots, x_j, \dots, x_n) & f(x_1, \dots, x_i, \dots, x'_j, \dots, x_n) \\ f(x_1, \dots, x'_i, \dots, x_j, \dots, x_n) & f(x_1, \dots, x'_i, \dots, x'_j, \dots, x_n) \end{vmatrix} \geq 0 \quad (2.3.3)$$

for every  $x_i < x'_i, x_j < x'_j$  ( $i = 1, \dots, n, j = 1, \dots, n$ ). He also showed that if (2.3.3) holds for a probability density function  $f$ , then every marginal of  $f$  satisfies the  $TP_2$  in pairs condition. However, if instead of (2.3.3) one has

$$\begin{vmatrix} f(x_1, \dots, x_i, \dots, x_j, \dots, x_n) & f(x_1, \dots, x_i, \dots, x'_j, \dots, x_n) \\ f(x_1, \dots, x'_i, \dots, x_j, \dots, x_n) & f(x_1, \dots, x'_i, \dots, x'_j, \dots, x_n) \end{vmatrix} \leq 0 \quad (2.3.4)$$

for every choice  $x_i < x'_i, x_j < x'_j$ , then it does not necessarily imply that every marginal of  $f$  satisfies (in pairs) something similar to (2.3.4).

The following example illustrates this.

Let  $X_1, X_2$  and  $X_3$  be identically independent random variables with common probability density function (pdf)  $f(x) = 2x, 0 \leq x \leq 1$ . Let  $Y_1 \leq Y_2 \leq Y_3$  denote the ordered  $X_i$ 's. Then joint pdf of  $Y_1, Y_2$  and  $Y_3$  is

$$g(y_1, y_2, y_3) = 48 y_1 y_2 y_3, \quad 0 \leq y_1 \leq y_2 \leq y_3 \leq 1. \quad (2.3.5)$$

It is easy to check from (2.3.5) that  $g(y_1, y_2, y_3)$  satisfies (2.3.4) in pairs. However, note that the joint pdf of  $Y_1$  and  $Y_3$  is

$$h(y_1, y_3) = 24 y_1 y_3 (y_3^2 - y_1^2), \quad 0 \leq y_1 \leq y_3 \leq 1. \quad (2.3.6)$$

Hence if  $y_1 < y'_1 < y_3 < y'_3$ , it follows from (2.3.6) that

$$\begin{aligned} & h(y_1, y_3) h(y'_1, y'_3) - h(y_1, y'_3) h(y'_1, y_3) \\ &= 576 y_1 y_3 y'_1 y'_3 [(y'_3{}^2 - y_1^2)(y_3^2 - y'_1{}^2) - (y'_3{}^2 - y'_1{}^2)(y_3^2 - y_1^2)] \\ &= 576 y_1 y_3 y'_1 y'_3 (y'_3{}^2 - y_3^2)(y_1^2 - y'_1{}^2) \geq 0, \end{aligned}$$

i. e. the joint pdf of  $Y_1$  and  $Y_3$  is  $TP_2$ .

We shall now show how conditions similar to (2.3.4) for every marginal leads to CDS.

**Theorem 14.** Let (a)  $f(x_1, \dots, x_n)$  denote the joint pdf of  $(X_1, \dots, X_n)$  satisfying (2.3.4) in every pairs of arguments when the remaining arguments are held fixed. It is also assumed that (b) all the marginals  $f_k(x_1, \dots, x_k), 1 \leq k < n$  satisfy an analogous version of (2.3.4) for

every pair of arguments when the remaining arguments are held fixed.

Then, (i)  $(X_1, \dots, X_n)$  is CDS; (ii) every permutation of  $X_1, \dots, X_n$  is CDS.

Proof. Fix  $x_3, \dots, x_n$  each at  $+\infty$ . Then,  $f_2(x_1, x_2)$  satisfies (2.3.2), so that by Theorem 13,  $(X_1, X_2)$  is NRD. Again for fixed  $x_2$ ,  $f_3(x_1, x_2, x_3)$  satisfies an analogous version of (2.3.4). Hence, for fixed  $x_2$ ,  $P(X_3 > x_3 | X_1 = x_1, X_2 = x_2)$  is  $\downarrow$  in  $x_1$  for fixed  $x_2$ . Repetition of this argument yield the desired result that  $P(X_i > x_i | X_1 = x_1, \dots, X_{i-1} = x_{i-1})$  is decreasing in  $x_1, \dots, x_{i-1}$  for each  $i = 2, \dots, n$ . By symmetry, every permutation of  $(X_1, \dots, X_n)$  is also CDS.

The next theorem proves RTDS property for a sequence of random variables when the tail of the distribution function satisfies properties similar to (2.3.4). Define  $\overline{F}_k(x_1, \dots, x_k) = P(X_1 > x_1, \dots, X_k > x_k)$ ,  $1 \leq k \leq n$ .

Theorem 15. Let (a)  $\overline{F}_n(x_1, \dots, x_n)$  satisfy (2.3.4) for every pair of arguments for fixed values of the remaining arguments; (b)  $\overline{F}_k(x_1, \dots, x_k)$  also satisfies the analogous version of (2.3.4) for every pair of arguments for fixed values of the remaining arguments. Then (i)  $(X_1, \dots, X_n)$  is RTDS; (ii) any permutation of  $(X_1, \dots, X_n)$  is also RTDS.

Proof. The proof of this theorem is very similar to that of Theorem

14, and hence is omitted.

The following lemma is needed to prove the next major theorem, namely the CDS property is preserved under limits.

Lemma 2. Let  $(X_1, \dots, X_n)$  be CDS. Then  $E(\psi(X_i) | X_1 = x_1, \dots, X_{i-1} = x_{i-1})$  is decreasing in  $x_1, \dots, x_{i-1}$  for every increasing integrable function  $\psi$ . Moreover  $X = (X_1, \dots, X_n)$  is CDS if and only if  $E(\psi(X_i) | X_1 = x_1, \dots, X_{i-1} = x_{i-1})$  is decreasing in  $x_1, \dots, x_{i-1}$  for all real valued bounded continuous increasing  $\psi$ .

Proof. The proof of this lemma is comparable to the proof of Theorem 7. Hence, it is omitted.

We are now in a position to prove the preservation theorem of CDS under limits.

Theorem 16. Let  $\{X_n, n \geq 1\}$  be a sequence of p-dimensional CDS random vectors with distribution function  $\{H_n, n \geq 1\}$  such that  $H_n \rightarrow H$  weakly as  $n \rightarrow \infty$  where  $H$  is the distribution function of a p-dimensional random vector  $X$ . Then,  $X$  is also CDS.

Proof. Use Lemma 2 and the Helly-Bray Theorem.

We now aim at showing that  $\text{CDS} \Rightarrow \text{NOD}$ . Ahmed et al.(1978b) have shown that  $\text{CIS} \Rightarrow \text{Associatedness} \Rightarrow \text{POD}$ . We find it hard to develop a meaningful dual of associatedness for negative dependence. Hence, we try to build a direct proof of the fact that  $\text{CDS} \Rightarrow \text{NOD}$ . With this end, we first prove the following lemma.

Lemma 3. Let  $(X_1, \dots, X_n)$  be CDS. Then  $X^{(\ell)} = (X_1, \dots, X_{\ell-1}, X_{\ell+1}, \dots, X_n)$  is CDS for each  $\ell = 1, \dots, n$ .

Proof. All we need show is for any  $j \geq \ell + 1$ ,

$P(X_j > x_j | X_1 = x_1, \dots, X_{i-1} = x_{i-1}, \dots, X_{j-1} = x_{j-1}; i \neq \ell)$  is  $\downarrow$  in  $x_1, \dots, x_i, \dots, x_{j-1} (i \neq \ell)$ . But for any nondecreasing real valued function  $\psi$ , if  $x_i < x'_i (i \neq \ell)$ , for  $j \geq \ell + 1$ ,

$$\begin{aligned} & E[\psi(X_j) | X_1 = x_1, \dots, X_i = x_i, \dots, X_{j-1} = x_{j-1}, i \neq \ell] \\ &= E_{X_\ell} E[\psi(X_j) | X_1 = x_1, \dots, X_i = x_i, \dots, X_{j-1} = x_{j-1}, X_\ell] \\ &\geq E_{X_\ell} E[\psi(X_j) | X_1 = x_1, \dots, X_i = x'_i, \dots, X_{j-1} = x_{j-1}, X_\ell] \\ &= E[\psi(X_j) | X_1 = x_1, \dots, X_i = x'_i, \dots, X_{j-1} = x_{j-1}; i \neq \ell]. \end{aligned}$$

This proves the lemma. We need the following immediate corollary to this lemma.

Corollary. Let  $\tilde{X}^* = (X_{i_1}, \dots, X_{i_k})$  be a  $k$ -dimensional subvector of  $\tilde{X} = (X_1, \dots, X_n)$ . Then  $\tilde{X}$  is CDS  $\Rightarrow \tilde{X}^*$  is CDS.

We are now in a position to prove the final main theorem of this section.

Theorem 17. Let  $\tilde{X} = (X_1, \dots, X_n)$  be CDS. Then  $(X_1, \dots, X_n)$  is NOD.

Proof. We use induction. For  $n = 2$ , the result follows from the fact (see Theorem 12) that  $\text{NRD} \Rightarrow \text{RTD} \Rightarrow \text{NQD}$ . Assume now the result to be true for  $n = k$ , and prove it for  $n = k + 1$ . First note that

$$\begin{aligned}
& P(X_{k+1} > x_{k+1}, \dots, X_2 > x_2 | X_1 = x) \\
&= P(X_{k+1} > x_{k+1} | X_k > x_k, \dots, X_2 > x_2, X_1 = x) \times \\
&P(X_k > x_k | X_{k-1} > x_{k-1}, \dots, X_2 > x_2, X_1 = x) \times \dots \times \\
&P(X_2 > x_2 | X_1 = x).
\end{aligned}$$

Using the CDS property of the given sequence of random variables, it follows that each of

$$\begin{aligned}
& P(X_{k+1} > x_{k+1} | X_k > x_k, \dots, X_2 > x_2, X_1 = x), \\
& P(X_k > x_k | X_{k-1} > x_{k-1}, \dots, X_2 > x_2, X_1 = x), \dots, \\
& P(X_2 > x_2 | X_1 = x) \text{ is } \downarrow \text{ in } x.
\end{aligned}$$

Writing now

$$Y = I_{[X_{k+1} > x_{k+1}, \dots, X_2 > x_2]}, \quad P(Y = 1 | X_1 = x) \text{ is } \downarrow \text{ in } x.$$

Hence,  $P(Y = 1, X_1 > x_1) \leq P(Y = 1) P(X_1 > x_1)$  i. e.

$$\begin{aligned}
& P(X_{k+1} > x_{k+1}, \dots, X_2 > x_2, X_1 > x_1) \\
&\leq P(X_{k+1} > x_{k+1}, \dots, X_2 > x_2) P(X_1 > x_1).
\end{aligned} \tag{2.3.7}$$

From the corollary to Lemma 3,  $(X_2, \dots, X_{k+1})$  is also CDS.

Hence, using the induction hypothesis

$$P(X_{k+1} > x_{k+1}, \dots, X_2 > x_2) \leq \prod_{i=2}^{k+1} P(X_i > x_i). \tag{2.3.8}$$

Combining (2. 3. 7) and (2. 3. 8),

$$P (X_{k+1} > x_{k+1}, \dots, X_2 > x_2, X_1 > x_1) \leq \prod_{i=1}^{k+1} P (X_i > x_i).$$

This proves the theorem.

The following lemma shows that CDS in fact implies the stronger SNOD property.

Lemma 4.  $(X_1, \dots, X_n)$  is CDS  $\Rightarrow (X_1, \dots, X_n)$  is SNOD.

Proof. In view of Theorem 17, it remains only to show that

$$P (X_1 \leq x_1, \dots, X_n \leq x_n) \leq \prod_{i=1}^n P (X_i \leq x_i)$$

for all real  $x_1, \dots, x_n$ . Let  $Y_i = -X_i$  ( $i = 1, \dots, n$ ). Note that  $(Y_1, \dots, Y_n)$  is also CDS. Hence,

$$\begin{aligned} P (X_1 \leq x_1, \dots, X_n \leq x_n) &= P (Y_1 \geq -x_1, \dots, Y_n \geq -x_n) \\ &\leq \prod_{i=1}^n P (Y_i \geq -x_i) = \prod_{i=1}^n P (X_i \leq x_i), \end{aligned}$$

for all real numbers  $x_1, \dots, x_n$ . The result follows.

## 2. 4. Applications

First we see an application in reliability theory. With this end, the following theorem is proved.

Theorem 18. If  $X_1, \dots, X_n$  are SNOD binary random variables, then,

$$(a) \quad P \left( \prod_{i=1}^n X_i = 1 \right) \leq \prod_{i=1}^n P (X_i = 1);$$

$$(b) \quad P\left(\bigcup_{i=1}^n X_i = 1\right) \geq \bigcup_{i=1}^n P(X_i = 1), \text{ where } \bigcup_{i=1}^n x_i = 1 - \prod_{i=1}^n (1 - x_i)$$

for all  $x_1, \dots, x_n$ .

Proof. (a)  $P\left(\prod_{i=1}^n X_i = 1\right) = P(X_1 = 1, \dots, X_n = 1) = P(X_1 > 0, \dots, X_n > 0)$

$$\leq \prod_{i=1}^n P(X_i > 0) = \prod_{i=1}^n P(X_i = 1);$$

$$(b) \quad P\left(\bigcup_{i=1}^n X_i = 1\right) = P\left(\max_{1 \leq i \leq n} X_i = 1\right) = 1 - P\left(\max_{1 \leq i \leq n} X_i = 0\right)$$

$$= 1 - P(X_1 \leq 0, \dots, X_n \leq 0) \geq 1 - \prod_{i=1}^n P(X_i \leq 0)$$

$$= 1 - \prod_{i=1}^n P(X_i = 0) = \bigcup_{i=1}^n P(X_i = 1)$$

In fact even if  $X_1, \dots, X_n$  are not binary but still SNOD, one gets the inequalities

$$P\left(\min_{1 \leq i \leq n} X_i > x\right) \leq \prod_{i=1}^n P(X_i > x) \quad (2.4.1)$$

$$P\left(\max_{1 \leq i \leq n} X_i > x\right) \geq \bigcup_{i=1}^n P(X_i > x) \quad (2.4.2)$$

Remark 8. The above theorem shows that if we calculate the reliability of a series system assuming the components to be independent when in fact they are NOD but not independent, we will overestimate system reliability. The reverse is true for a parallel system.



Next we show how a few well-known multivariate distributions exhibit some negative dependence among component variables.

Example 1. Let  $(X_1, \dots, X_n)$  have a multinormal distribution with mean vector  $(\mu_1, \dots, \mu_n)$  and variance-covariance matrix  $\Sigma$  which is positive definite. Let  $R = ((r_{ij})) = \Sigma^{-1}$ . Suppose  $r_{ij} > 0$  for all  $1 \leq i, j \leq n$ . Since the joint pdf of  $(X_1, \dots, X_n)$  is

$$f(x_1, \dots, x_n) = (2\pi)^{-\frac{n}{2}} |\Sigma|^{-\frac{1}{2}} \exp \left[ -\frac{1}{2} \sum_{1 \leq i < j \leq n} r_{ij} (x_i - \mu_i)(x_j - \mu_j) \right]$$

for any pair  $(i, j)$ , one can write

$$f(x_1, \dots, x_n) = c_1^{(i)} c_2^{(j)} \exp(-r_{ij} x_i x_j),$$

where  $\tilde{x}^{(i)} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ ,  $1 \leq i \leq n$ . Now for any

$x_i < x'_i, x_j < x'_j$  ( $i < j$ ) one has

$$\begin{aligned} & \left| \begin{array}{cc} f(x_1, \dots, x_i, \dots, x_j, \dots, x_n) & f(x_1, \dots, x'_i, \dots, x'_j, \dots, x_n) \\ f(x_1, \dots, x'_i, \dots, x_j, \dots, x_n) & f(x_1, \dots, x'_i, \dots, x'_j, \dots, x_n) \end{array} \right| \\ &= c_1^{(i)} c_1^{(i)} c_2^{(j)} c_2^{(j')} \{ \exp[-r_{ij}(x_i x_j + x'_i x'_j)] \\ & \quad - \exp[-r_{ij}(x_i x'_j + x'_i x_j)] \} < 0. \end{aligned} \quad (2.4.3)$$

Since  $(x_i x_j + x'_i x'_j) - (x_i x'_j + x'_i x_j) = (x'_i - x_i)(x'_j - x_j) > 0$  and  $r_{ij} > 0$ .

Also for any subset  $(X_{i1}, \dots, X_{ik})$  of  $(X_1, \dots, X_n)$ , an inequality similar to (2.4.3) is satisfied. Hence, using Theorem 14,  $(X_1, \dots, X_n)$

is CDS, and hence SNOD.

Example 2. Let  $\underline{X} = (X_1, \dots, X_k)$  have a multinomial distribution with joint probability function

$$f(x_1, \dots, x_k) = \frac{N!}{\prod_{i=1}^k x_i! (N - \sum_{i=1}^k x_i)!} \prod_{i=1}^k p_i^{x_i} (1 - \sum_{i=1}^k p_i)^{N - \sum_{i=1}^k x_i},$$

$$x_i \geq 0, \sum_{i=1}^k x_i \leq N, p_i \geq 0, \sum_{i=1}^k p_i \leq 1.$$

Note also that any subset  $(X_{i1}, \dots, X_{i\ell})$  of  $(X_1, \dots, X_k)$  is multinomial with appropriate parameters. Now for any  $x_i < x'_i, x_j < x'_j$ ,  $i < j$ ,

$$\begin{aligned} & \left| \frac{f(x_1, \dots, x_i, \dots, x_j, \dots, x_k)}{f(x_1, \dots, x'_i, \dots, x_j, \dots, x_k)} - \frac{f(x_1, \dots, x_i, \dots, x'_j, \dots, x_k)}{f(x_1, \dots, x'_i, \dots, x'_j, \dots, x_k)} \right| \\ &= \frac{(N!)^2}{(\prod_{\ell \neq i \neq j} x_\ell!)^2} \left( \prod_{\ell \neq i \neq j} p_\ell^{x_\ell} \right)^2 p_i^{x_i + x'_i} p_j^{x_j + x'_j} \\ & \times (1 - \sum_{i=1}^k p_i)^{2N - 2 \sum_{\ell \neq i \neq j} x_\ell - (x_i + x'_i) - (x_j + x'_j)} \\ & \times \left[ \{(N - \sum_{\ell \neq i \neq j} x_\ell - x_i - x_j)!\}^{-1} \{(N - \sum_{\ell \neq i \neq j} x_\ell - x'_i - x'_j)!\}^{-1} \right. \\ & \left. - \{(N - \sum_{\ell \neq i \neq j} x_\ell - x_i - x'_j)!\}^{-1} \{(N - \sum_{\ell \neq i \neq j} x_\ell - x'_i - x_j)!\}^{-1} \right] \leq 0. \end{aligned}$$

A similar inequality holds for the joint probability function of any subset of  $(X_1, \dots, X_k)$ , and once again, from Theorem 14,  $(X_1, \dots, X_k)$  is CDS, and hence SNOD.

Example 3. Let  $X_1, \dots, X_k$  have the joint Dirichlet pdf

$$f(x_1, \dots, x_k) = \frac{\Gamma(\sum_{i=1}^k a_i)}{\prod_{i=1}^k \Gamma(a_i)} \prod_{i=1}^k x_i^{a_i-1} (1 - \sum_{i=1}^k x_i)^{a_{k+1}-1},$$

$$x_i > 0, \sum_{i=1}^k x_i < 1; a_i > 0 \ (1 \leq i \leq k), \ a_{k+1} \geq 1.$$

Now for  $x_i < x'_i, x_j < x'_j$ , one has

$$\begin{aligned} & \left| \begin{array}{cc} f(x_1, \dots, x_i, \dots, x_j, \dots, x_k) & f(x_1, \dots, x_i, \dots, x'_j, \dots, x_k) \\ f(x_1, \dots, x'_i, \dots, x_j, \dots, x_k) & f(x_1, \dots, x'_i, \dots, x'_j, \dots, x_k) \end{array} \right| \\ & \propto \prod_{\ell \neq i \neq j} p_\ell^{2x_\ell} p_i^{x_i + x'_i} p_j^{x_j + x'_j} \\ & \times \left[ \left\{ (1 - \sum_{\ell \neq i \neq j} (x_\ell - x_i - x_j)) (1 - \sum_{\ell \neq i \neq j} x_\ell - x'_i - x'_j) \right\}^{a_{k+1}-1} \right. \\ & \left. - \left\{ (1 - \sum_{\ell \neq i \neq j} x_\ell - x_i - x'_j) (1 - \sum_{\ell \neq i \neq j} x_\ell - x'_i - x_j) \right\}^{a_{k+1}-1} \right]. \quad (2.4.4) \end{aligned}$$

Note that for any  $x_i < x'_i, x_j < x'_j$ ,

$$(1 - \sum_{\ell \neq i \neq j} x_\ell - x_i - x_j) (1 - \sum_{\ell \neq i \neq j} x_\ell - x'_i - x'_j)$$

$$\begin{aligned}
& - (1 - \sum_{\ell \neq i \neq j} x_{\ell} - x_i - x_j) (1 - \sum_{\ell \neq i \neq j} x_{\ell} - x_i' - x_j') \\
& = x_i x_j' + x_i' x_j - x_i' x_i - x_j' x_j = - (x_i' - x_i) (x_j' - x_j) < 0.
\end{aligned}$$

Since  $a_{k+1} \geq 1$ , the right hand side of (2.4.4)  $\leq 0$ . Since, any subset of Dirichlet random variables also has a Dirichlet distribution, it follows from Theorem 14 that  $(X_1, \dots, X_k)$  is CDS and hence SNOD.

In all the above three examples, in the bivariate case, the two random variables are mutually NRD, and hence these are NQD.

Example 4. A general class of negatively dependent multivariate distributions is given by the Farlie-Gumbel-Morgenstern (see Johnson and Kotz (1975)) system in some cases.

Consider  $n$  random variables  $X_1, \dots, X_n$  with joint pdf of the form

$$\begin{aligned}
f(x_1, \dots, x_n) &= \prod_{j=1}^n f_j(x_j) \left[ 1 + \sum_{1 \leq j_1 < j_2 \leq n} a_{j_1, j_2} \right. \\
&\quad \left. (1 - 2 F_{j_1}(x_{j_1})) (1 - 2 F_{j_2}(x_{j_2})) \right. \\
&\quad \left. + \dots + a_{12 \dots n} \prod_{j=1}^n (1 - 2 F_j(x_j)) \right], \tag{2.4.5}
\end{aligned}$$

where the  $F_j$ 's are distribution functions with corresponding pdf's  $f_j$ 's and the constant  $a$ 's satisfy certain conditions (see Johnson and Kotz (1975, section 4) for constraints on the  $a$ 's).

The system of distributions defined in (2.4.5) is known as the Farlie-Gumbel-Morgenstern (FGM) system for  $n = 2$ . Johnson and Kotz (1975) provide the multivariate generalization of this system.

For  $n = 2$ , in order that  $f(x_1, x_2)$  is a pdf,  $|a_{12}|$  must be less than or equal to 1. It is easy to check that  $(X_1, X_2)$  is NQD if and only if  $-1 \leq a_{12} < 0$ .

We see now that for  $n = 2$ ,  $f$  defined in (2.4.5) in fact satisfies (2.3.4) and so  $X_1$  and  $X_2$  are mutually NRD, and hence NQD.

With this end, first write

$$f(x_1, x_2) = f_1(x_1)f_2(x_2) [1 + a_{12}(1 - 2F_1(x_1))(1 - 2F_2(x_2))].$$

Hence for  $x_1 < x'_1, x_2 < x'_2$ ,

$$\begin{aligned} & \begin{vmatrix} f(x_1, x_2) & f(x_1, x'_2) \\ f(x'_1, x_2) & f(x'_1, x'_2) \end{vmatrix} \\ &= a_{12} f_1(x_1)f_1(x'_1)f_2(x_2)f_2(x'_2) [(1 - 2F_1(x_1))(1 - 2F_2(x_2)) \\ & \quad + (1 - 2F_1(x'_1))(1 - 2F_2(x'_2)) - (1 - 2F_1(x'_1))(1 - 2F_2(x_2)) \\ & \quad - (1 - 2F_1(x_1))(1 - 2F_2(x'_2))] \\ &= 4a_{12} f_1(x_1)f_1(x'_1)f_2(x_2)f_2(x'_2) [F_1(x'_1) - F_1(x_1)] \\ & \quad [F_2(x'_2) - F_2(x_2)] \leq 0, \end{aligned}$$

since  $x_1 < x'_1, x_2 < x'_2$ ,  $F_j$ 's are distribution functions and  $a_{12} \leq 0$ .

Johnson and Kotz (1975, section 5) have provided necessary and sufficient conditions for the SNOD property for the FGM system. We demonstrate now that these conditions do not imply in general the property (2.3.4) for  $n \geq 3$ .

Consider the case  $n = 3$ . Johnson and Kotz (1975) have shown that in this case  $(X_1, X_2, X_3)$  is SNOD (negative F and G orthant dependent in their terminology) if and only if

$$\begin{aligned} a_{ij} &\leq 0 \quad (1 \leq i < j \leq 3), \quad a_{12} + a_{13} + a_{23} + a_{123} \leq 0, \\ a_{12} + a_{13} + a_{23} - a_{123} &\leq 0. \end{aligned} \quad (2.4.6)$$

Now if  $x_2 < x'_2, x_3 < x'_3$ , keeping  $x_1$  fixed,

$$\begin{aligned} &\begin{vmatrix} f(x_1, x_2, x_3) & f(x_1, x'_2, x_3) \\ f(x_1, x_2, x'_3) & f(x_1, x'_2, x'_3) \end{vmatrix} \\ &= \{a_{23} + a_{123}(1 - 2F_1(x_1)) - a_{12}a_{13}(1 - 2F_1(x_1))^2\} \\ &\quad \times f_1^2(x_1)f_2(x_2)f_2(x'_2)f_3(x_3)f_3(x'_3) \\ &\quad \times [(1 - 2F_2(x_2))(1 - 2F_3(x_3)) + (1 - 2F_2(x'_2))(1 - 2F_3(x'_3)) \\ &\quad - (1 - 2F_2(x_2)) \times (1 - 2F_3(x'_3)) - (1 - 2F_2(x'_2))(1 - 2F_3(x_3))] \\ &= [(a_{23} + a_{123} - a_{12}a_{13}) + 2F_1(x_1)(2a_{12}a_{13} - a_{123}) \\ &\quad - 4a_{12}a_{13}F_1^2(x_1)] \end{aligned}$$

$$\begin{aligned} & \times [F_2(x'_2) - F_2(x_2)] [F_3(x'_3) - F_3(x_3)] \\ & \times f_1^2(x_1) f_2(x_2) f_2(x'_2) f_3(x_3) f_3(x'_3). \end{aligned} \quad (2.4.7)$$

Assume  $F_1$  is a continuous distribution function. Taking  $x_1$  such that  $F_1(x_1) = \frac{3}{4}$ , one gets the right hand side of (2.4.7)  $= (a_{23} - \frac{1}{2}a_{123} - \frac{1}{4}a_{12}a_{13}) \times$  (a nonnegative quantity). Now with the choice  $a_{12} = a_{13} = a_{23} = -0.1$ ,  $a_{123} = -0.25$ , it follows that (2.4.6) is satisfied, but

$$a_{23} - \frac{1}{2}a_{123} - \frac{1}{4}a_{12}a_{13} = .025 - .0025 > 0. \quad (2.4.8)$$

Thus (2.3.4) does not necessarily hold when  $x_1$  is held fixed. From the symmetry in the choice of  $a_{12}, a_{13}$  and  $a_{23}$ , it follows that a similar conclusion holds, when either  $x_2$  or  $x_3$  is held fixed.

Note also that when (2.4.6) holds, the sequence  $(X_1, X_2, X_3)$  is not necessarily RTDS. To see this, first write

$$\begin{aligned} P(X_3 > x_3 | X_1 > x_1, X_2 > x_2) &= P(X_3 > x_3) \left[ 1 \right. \\ & \left. + \frac{a_{13}F_1(x_1)F_3(x_3) + a_{23}F_2(x_2)F_3(x_3) - a_{123}\prod_{i=1}^3 F_i(x_i)}{1 + a_{12}F_1(x_1)F_2(x_2)} \right]. \end{aligned} \quad (2.4.9)$$

Assume each  $F_i$  is continuous function. Choose  $x_1, x_3$  such that  $F_1(x_1) = \frac{1}{2}$ ,  $F_3(x_3) > 0$ . Also, let  $a_{12} = -0.3$ ,  $a_{13} = -0.1$ ,  $a_{23} = -0.1$  and  $a_{123} = -0.4$  so that (2.4.6) is satisfied. Now with

the choice  $x_2' < x_2''$  such that  $F_2(x_2') = \frac{1}{2}$ ,  $F_2(x_2'') = \frac{3}{4}$ , it follows that the expression in the brackets of (2.4.9) with  $x_2 = x_2'$  is 1, while the expression in the brackets of (2.4.9) with  $x_2 = x_2''$  is bigger than 1. This shows that the RTDS property does not necessarily hold.

Our next application is comparable to the result of Jogdeo 1968 (see also Ahmed et al. (1978b) for a simple proof) which is relevant in characterizing independence in  $2 \times 2$  contingency tables. The corresponding POD counterpart is proved in Jogdeo (1968).

Theorem 19. Let  $(X_1, X_2, X_3)$  be NOD. Then  $X_1, X_2$  and  $X_3$  are independent if and only if (i)  $\text{COV}(X_i, X_j) = 0$   $1 \leq i < j \leq 3$ ; (ii) any one pair, say  $(X_1, X_2)$  satisfies  $E(X_1 X_2 | X_3) = E(X_1 | X_3) \cdot E(X_2 | X_3)$ .

Proof. Since  $(X_1, X_2, X_3)$  is NOD, any pair  $(X_i, X_j)$  is NQD.

Hence

$$P(X_i \leq x, X_j \leq y) \leq P(X_i \leq x) P(X_j \leq y) \text{ for all real } x \text{ and } y.$$

Use the Hoeffding identity (see for example Lehmann (1966)).

$$\begin{aligned} \text{COV}(X_i, X_j) &= \iint [P(X_i \leq x, X_j \leq y) \\ &\quad - P(X_i \leq x) P(X_j \leq y)] dx dy \end{aligned} \quad (2.4.10)$$

In view of (i)  $\text{COV}(X_i, X_j) = 0$ . Also, the integrand in (2.4.10) being nonpositive, one must have  $P(X_i \leq x, X_j \leq y) = P(X_i \leq x) P(X_j \leq y)$  for all real  $x$  and  $y$ , which ensures pairwise independence.

Now, since  $(X_1, X_2, X_3)$  is NOD, one gets



$$P(X_1 > x_1, X_2 > x_2 | X_3 > x_3) \leq P(X_2 > x_2) P(X_1 > x_1). \quad (2.4.11)$$

Note that an alternate representation of (2.4.11) is

$$P(X_1 > x_1, X_2 > x_2 | X_3 > x_3) \leq P(X_2 > x_2 | X_3 > x_3) \\ P(X_1 > x_1 | X_3 > x_3) \quad (2.4.12)$$

using pairwise independence of  $(X_1, X_3)$  and  $(X_2, X_3)$ . Since (ii) ensures the conditional uncorrelation of  $X_1$  and  $X_2$  given  $X_3$ , using the Hoeffding identity again, it follows that the two sides of (2.4.12) are equal, that is the two sides of (2.4.11) must be equal. This ensures  $P(X_1 > x_1, X_2 > x_2, X_3 > x_3) = \prod_{i=1}^3 P(X_i > x_i)$ . The theorem follows.

Finally, an important class of distributions satisfying the NQD property is furnished below. This class is essentially an extension of a similar class considered by Lehmann (1966). Before stating the main result, we introduce the following definition.

Definition 2.4.1. Two functions  $g_1$  and  $g_2$  of  $n$  arguments are said to be concordant if considered as a function of the  $i$ th coordinate (with all other coordinates held fixed), they are monotone in the same direction, (that is either both increasing or both decreasing).

Theorem 20. Let  $(X_i, Y_i) | w$  be independent random variables satisfying the conditions of Theorem 1 with joint distributions  $F_1^w, \dots, F_n^w$  respectively. Let  $g_1, g_2: R^n \rightarrow R$  be concordant functions. Then

$Z_1 = g_1(X_1, \dots, X_n)$ ,  $Z_2 = g_2(Y_1, \dots, Y_n)$  are NQD.

Proof. Use Theorem 1 and Lehmann's Theorem 1.

Corollary. If  $F$  satisfies the hypothesis of Theorem 20, then Kendall's  $\tau$ , Spearman's  $\rho_s$  and the quadrant measure  $q$  discussed by Blomquist (1950) are all nonpositive.

Proof. (i) Since Kendall's  $\tau$  is the correlation coefficient of  $X = \text{Sgn}(X_2 - X_1)$  and  $Y = \text{Sgn}(Y_2 - Y_1)$ , the result follows from Theorem 20.

(ii) Since Spearman Rho is the correlation coefficient of  $X = \text{Sgn}(X_2 - X_1)$  and  $Y = \text{Sgn}(Y_3 - Y_1)$ , the result follows from Theorem 20.

(iii) Let  $\mu_1$  and  $\mu_2$  denote the medians of the marginal distributions of  $X$  and  $Y$ , and let  $f(X)$ ,  $g(Y)$  indicate the events  $X > \mu_1$  and  $Y > \mu_2$  respectively. Then

$$q = E[fg + (1-f)(1-g) - f(1-g) - g(1-f)] = E[1 + 4fg - 2f - 2g].$$

By Theorem 20,  $E(fg) \leq E(f)E(g)$  and hence

$$q \leq [1 - 2E(f)][1 - 2E(g)] \leq 0.$$

Consider now testing the hypothesis  $H$  against a number of different sets of alternatives  $K_i$  ( $i = 1, \dots, n$ ). Suppose  $H$  is rejected in favor of  $K_i$  if

$$T_i \geq C_i \tag{2.4.13}$$

at level  $\alpha_i$  and that the tests are similar so that

$$P_H(T_i \geq C_i) = \alpha_i \quad (2.4.14)$$

Typical examples are the so-called slippage problems in which it is assumed of  $k$  parameters  $\theta_1, \dots, \theta_k$  that they are either all equal ( $H$ ) or that exactly one of them has "slipped", i. e. is different from others,  $K_i$  denoting the event that  $\theta_i$  has slipped. In such situations, rather than controlling the individual error probabilities (2.4.14), it is frequently of interest to control the experiment wise error, that is, the probability  $P$  of falsely rejecting  $H$  in favor of any of the alternatives  $K_i$ . Applying Bonferroni's inequalities to the (2.4.13), we obtain for  $P$  the inequalities

$$\sum \alpha_i - \sum_{i < j} P(T_i \geq C_i, T_j \geq C_j) \leq P \leq \sum \alpha_i. \quad (2.4.15)$$

Suppose that  $(T_1, \dots, T_k)$  is NQD, so that

$$P(T_i \geq C_i, T_j \geq C_j) \leq \alpha_i \alpha_j. \quad (2.4.16)$$

Then it follows from (2.4.15) that

$$\sum \alpha_i - \sum_{i < j} \alpha_i \alpha_j \leq P \leq \sum \alpha_i. \quad (2.4.17)$$

To see how close these bounds are, note that if  $\alpha = \sum \alpha_i$ , then

$$\sum_{i < j} \alpha_i \alpha_j \leq \frac{1}{2} (1 - k^{-1}) \alpha^2 \text{ so that}$$

$$\alpha - \frac{1}{2} \alpha^2 (1 - k^{-1}) \leq P \leq \alpha. \quad (2.4.18)$$

This shows  $\alpha$  to be a satisfactory approximation for  $P$  whenever it is small.

### 3. THE ORDERING OF NEGATIVE QUADRANT DEPENDENCE

#### 3.1. Ordering of NQD Random Vectors

The notions of positive and negative quadrant dependence (PQD and NQD) in the bivariate case were introduced in Chapters 1 and 2 respectively. Both PQD and NQD are qualitative forms of dependence, and indicate whether or not a pair of random variables exhibits positive or negative dependence. However, for many purposes, in addition to the knowledge of the nature of dependence, it is also important to know the degree of PQD or NQD-ness. Ahmad et al. (1979) have studied very extensively the partial ordering of PQD which permits us to compare pairs of PQD bivariate random vectors of interest with specified marginals as to their degree of PQD-ness. Quite in the same spirit, we study the degree of NQD-ness in this chapter.

Let  $\beta = \beta(F, G)$  denote the class of bivariate distribution functions  $H$  on  $R^2$  having specified marginal distribution functions  $F$  and  $G$ ,  $F$  and  $G$  being nondegenerate. Use the notation  $\bar{H}(x, y) = P(X > x, Y > y)$ .

Let  $\bar{\beta}$  denote the subclass of  $\beta$  where  $H$  is NQD. Suppose  $H_1$  and  $H_2$  both belong to  $\bar{\beta}$ . We then have the following definition.

Definition 3.1.1. The bivariate distribution  $H_2$  is said to be more negatively quadrant dependent than  $H_1$  if both  $H_1$  and  $H_2$  are NQD and

$$\bar{H}_2(x, y) \leq \bar{H}_1(x, y) \quad (3.1.1)$$

for all  $(x, y) \in \mathbb{R}^2$ . We write  $H_2 \xrightarrow{\text{NQD}} H_1$ .

Remark 1. Note that the requirement of specified marginals is important because otherwise we can alter the degree of NQD-ness by changing the scale.

Remark 2. An equivalent form of (3.1.1) is

$$H_2(x, y) \leq H_1(x, y) \quad (3.1.2)$$

for all  $(x, y) \in \mathbb{R}^2$ . Since  $H_1$  and  $H_2$  are NQD, it is clear that the distribution  $H_0$  belonging to  $\bar{\beta}$  exhibiting the least degree of NQD-ness is given by

$$H_0(x, y) = F(x) G(y) \quad (3.1.3)$$

for all  $(x, y) \in \mathbb{R}^2$ .

Next we demonstrate that there exists a pair  $(X^*, Y^*)$  distributed according to  $H^* \in \bar{\beta}$  such that  $Y^* = h(X^*)$  for some real valued non-increasing  $h$ .

First we need some preliminaries to introduce the distribution function

$$U^*(x, y) = \begin{cases} x + y - 1 & \text{if } 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad x + y \geq 1 \\ 1 & \text{if } x \geq 1, \quad y \geq 1 \\ 0 & \text{if } x + y < 1 \end{cases} \quad (3.1.4)$$

Then  $U^*(x, y)$  is singular bivariate distribution function with the uniform  $[0, 1]$  distribution as the marginals. Also,  $U^*$  concentrates all its mass on the line  $x + y = 1$ . By using the inequality

$$P(X \leq x, Y \leq y) \geq \max(0, P(X \leq x) + P(Y \leq y) - 1) \quad (3.1.5)$$

it follows that for any bivariate distribution  $H$  on  $I_2 = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$  with uniform  $[0, 1]$  marginals

$$H(x, y) \geq x + y - 1 \quad (3.1.6)$$

Next note that if  $F$  and  $G$  are continuous,  $F(X)$  and  $G(Y)$  are uniform  $[0, 1]$  random variables. If  $F$  or  $G$  admit discontinuities, suppose  $x(y)$  belongs to an open interval on which  $F(G)$  is constant. Let  $\tilde{x}(\tilde{y})$  denote the largest point in the support of  $F(G)$  with  $\tilde{x} < x$  ( $\tilde{y} < y$ ), then replace  $U^*(x, y)$  by  $\tilde{U}^*(x, y)$  with  $x$  and  $y$  by  $\tilde{x}$  and  $\tilde{y}$  in (3.1.4).

Then

$$P[F(X) \leq u] = P[G(Y) \leq u] = \begin{cases} \tilde{u} & \text{if } 0 < u < 1 \\ 0 & \text{if } u < 0 \\ 1 & \text{if } u \geq 1 \end{cases} \quad (3.1.7)$$

We are now in a position to prove the main theorem of this section.

Theorem 1. Let  $(X, Y)$  be continuous (not necessarily continuous) bivariate random variable distributed according to  $H \in \bar{\beta}$ . A necessary and sufficient condition that there exist almost surely (a.s.) a strictly decreasing (nonincreasing) function  $h$  such that  $Y = h(X)$  is that  $H$  equals  $H^*$  where

$$H^*(x, y) = \max(0, F(x) + G(y) - 1) \quad (3.1.8)$$

Proof. Let  $\tau$  be the class of all rectangles  $S_0$  in  $R^2$  such that  $S_0 = \{(x, y) | x(x', y') : x < x' \text{ and } y < y', F(x) = F(x') \text{ or } G(y) = G(y')\}$ , and let  $\tau_0$  denote the union of all  $S_0$  belonging to  $\tau$ . Then any  $H \in \bar{\beta}$  assigns no mass to  $\tau_0$ .

Now suppose  $Y = h(X)$  and  $h$  is a.s. a decreasing real valued function. We have outside a set of zero  $F$ -measure

$$F(x) = P(X \leq x) = P(Y \geq \sup_{y=h(x)} y) = 1 - G(\sup_{y=h(x)} y)$$

Hence

$$1 - G(Y) = 1 - G(h(X)) = F(X) \text{ a.s.,}$$

$$P\{F(X) \leq v, G(Y) \leq w\} = P\{F(X) \leq v, F(X) \geq 1 - w\}$$

$$= \begin{cases} 0 & \text{if } v < 0 \text{ or } w < 0 \\ v + w - 1 & \text{if } 0 \leq v \leq 1, 0 \leq w \leq 1, v + w \geq 1 \\ 1 & \text{if } 1 \leq v \text{ and } 1 \leq w \end{cases}$$



$$= P(F(X) \leq v, G(Y) \leq w),$$

and outside  $\tau_0$ ,

$$P(X \leq x, Y \leq y) = P\{F(X) \leq F(x), G(Y) \leq G(y)\}$$

$$= \begin{cases} F(x) + G(y) - 1 & \text{if } F(x) + G(y) \geq 1 \\ 0 & \text{if } F(x) + G(y) < 1 \\ 1 & \text{if } F(x) + G(y) \geq 2 \end{cases} \quad (3.1.9)$$

By definition of  $\tau_0$ , (3.1.4) holds in  $\tau_0$ .

Conversely suppose  $(X^*, Y^*)$  has the distribution  $H^*$ . Then

$(F(X^*), G(Y^*))$  has the distribution  $\tilde{U}^*$  i.e.,

$$U^*(F(x), G(y)) = \begin{cases} F(\tilde{x}) + G(\tilde{y}) - 1 & \text{if } F(\tilde{x}) + G(\tilde{y}) \geq 1 \\ 0 & \text{if } F(\tilde{x}) + G(\tilde{y}) < 1 \\ 1 & \text{if } F(\tilde{x}) + G(\tilde{y}) \geq 2 \end{cases}$$

implies that  $P\{(\tilde{x}, \tilde{y}) : F(\tilde{x}) + G(\tilde{y}) = 1\} = 1$ . Hence  $X$  is a non-increasing function of  $Y$ .

Remark 3. The theorem implies that among all pairs of random variables with prescribed marginals those which exhibit the most NQD-ness in the sense of (3.1.1) are exactly those which are nonincreasing functions of each other.

### 3.2. NQD Increasing in Parameter

In this section we have considered a family of bivariate distributions with specified marginals, the members of the family depending on a certain parameter, say  $\lambda$ .

Definition 3.2.1. A family of distributions  $H = \{H_\lambda(x, y) : \lambda \in \Lambda\}$ ,  $\Lambda \subset \mathbb{R}$ , is said to be increasingly negative quadrant dependent (decreasingly negative quadrant dependent) in  $\lambda$  if and only if

$$\lambda' > \lambda \Rightarrow H_{\lambda'} \xrightarrow{\text{NQD}} H_\lambda \quad (H_\lambda \xrightarrow{\text{NQD}} H_{\lambda'}).$$
(3.2.1)

Next we provide examples of certain families of distributions which are increasingly negative quadrant dependent or decreasingly negative quadrant dependent in the indexing parameter.

Example 3.1.1. Consider the bivariate Farlie-Gumbel-Morgenstern family of distributions with distribution function

$$H_a(x, y) = F(x) G(y) \{1 + a \bar{F}(x) \bar{G}(y)\} \quad -1 < a \leq 0. \quad (3.2.2)$$

Then, for  $-1 < a_1 < a_2 \leq 0$ ,

$$H_{a_1}(x, y) - H_{a_2}(x, y) = (a_1 - a_2) F(x) \bar{F}(x) G(y) \bar{G}(y) < 0.$$

Hence the above family of distributions is decreasing NQD in  $a$ .

Example 3.2.2. Let  $\Gamma$  be the class of distributions  $H_a$ ,

$$H_a = (1 - a) H_0 + a H^* \quad 0 \leq a \leq 1, \quad (3.2.3)$$

where  $H^*(x, y)$  is defined in (3.1.8).

From Chapter 2, we know that every convex combination of two bivariate NQD, which has fixed marginals is also NQD. Now,

$$H_a(x, y) = \begin{cases} (1 - a) F(x) G(y) & \text{if } F(x) + G(y) \leq 1 \\ F(x) + G(y) - a \{F(x) G(y) - F(x) - G(y) + 1\} & \\ \text{if } F(x) + G(y) \geq 1. & \end{cases} \quad (3.2.4)$$

Then, for  $a_1 < a_2$ ,

$$H_{a_1}(x, y) - H_{a_2}(x, y) = \begin{cases} (a_2 - a_1) F(x) G(y) & \text{if } F(x) + G(y) \leq 1 \\ (a_2 - a_1) \bar{F}(x) \bar{G}(y) & \text{if } F(x) + G(y) \geq 1 \end{cases}$$

so that  $H_{a_1}(x, y) - H_{a_2}(x, y) > 0$  for all  $0 \leq a_1 < a_2 \leq 1$ . Thus the families of distribution  $\Gamma$  is increasingly NQD in  $a$ .

Example 3.2.3. Let  $X = (X_1, X_2)'$  have density

$$g_{\sigma_{12}}(\underline{x}) = |\underline{\Sigma}|^{-\frac{1}{2}} f(\underline{x}' \underline{\Sigma}^{-1} \underline{x}), \quad (3.2.5)$$

where  $\underline{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$ . Then the family of distributions  $g_{\sigma_{12}}(\underline{x})$  with

$\sigma_1$  and  $\sigma_2$  fixed is decreasingly NQD in  $\sigma_{12} (\leq 0)$ , that is if  $\underline{Y} =$

$(Y_1, Y_2)'$  has the same form of density with  $\underline{\Sigma}^* = \begin{pmatrix} \sigma_1^2 & \sigma'_{12} \\ \sigma'_{12} & \sigma_2^2 \end{pmatrix}$  such

that  $\sigma_{12} < \sigma'_{12} \leq 0$ , then  $P(X_1 \leq x_1, X_2 \leq x_2) \leq P(Y_1 \leq x_1, Y_2 \leq x_2)$  for all  $x_1, x_2$  in  $R^2$ . The proof of the above result is given in Dasgupta and Olkin (1972). The bivariate normal with nonpositive correlation coefficient is a special case of this example.

Example 3.2.4. Let  $X$  and  $Y$  be independent random variables each having a continuous distribution function. Define  $V = X$  and  $W_\lambda = \lambda X + (1 - \lambda^2)^{\frac{1}{2}} Y$ ,  $-1 < \lambda \leq 0$ . If  $W_\lambda$  and  $W_{\lambda'}$  have the same marginal distribution, one gets the equality

$$\int_{-\infty}^{\infty} \bar{G} \left( \frac{y - \lambda x}{\sqrt{1 - \lambda^2}} \right) dF(x) = \int_{-\infty}^{\infty} \bar{G} \left( \frac{y - \lambda' x}{\sqrt{1 - \lambda'^2}} \right) dF(x) \quad (3.2.6)$$

for all  $y$ , where  $F$  and  $G$  denote the respective distribution functions of  $X$  and  $Y$ . Now,

$$P_\lambda (W > w, V > v) = \int_v^\infty \bar{G} \left( \frac{y - \lambda x}{\sqrt{1 - \lambda^2}} \right) dF(x) \quad (3.2.7)$$

Now, since  $\frac{d}{d\lambda} [\lambda(1 - \lambda^2)^{-\frac{1}{2}}] = (1 - \lambda^2)^{-\frac{3}{2}} \geq 0$  for  $-1 < \lambda \leq 0$ ,

it follows that for  $-1 < \lambda < \lambda' \leq 0$ ,  $\lambda(1 - \lambda'^2)^{\frac{1}{2}} < \lambda'(1 - \lambda^2)^{\frac{1}{2}}$ . Hence,

for  $-1 < \lambda < \lambda' \leq 0$ ,  $(w - \lambda x)(1 - \lambda^2)^{-\frac{1}{2}} < (w - \lambda' x)(1 - \lambda'^2)^{-\frac{1}{2}}$

$$\Leftrightarrow x \{ \lambda'(1 - \lambda'^2)^{\frac{1}{2}} - \lambda(1 - \lambda'^2)^{\frac{1}{2}} \} < w [ (1 - \lambda^2)^{\frac{1}{2}} - (1 - \lambda'^2)^{\frac{1}{2}} ]$$

$$\Leftrightarrow x < u = u(w, \lambda, \lambda') \text{ for some } u. \quad (3.2.8)$$

Thus for  $-1 < \lambda < \lambda' \leq 0$ ,

$$\bar{G}\left(\frac{w - \lambda x}{\sqrt{1 - \lambda^2}}\right) \leq \bar{G}\left(\frac{w - \lambda' x}{\sqrt{1 - \lambda'^2}}\right) \Leftrightarrow x \geq u. \quad (3.2.9)$$

Now if  $v > u$ , using (3.2.6)

$$\int_v^\infty \bar{G}\left(\frac{w - \lambda x}{\sqrt{1 - \lambda^2}}\right) dF(x) \leq \int_v^\infty \bar{G}\left(\frac{w - \lambda' x}{\sqrt{1 - \lambda'^2}}\right) dF(x) \quad (3.2.10)$$

While if  $v \leq u$ ,

$$\begin{aligned} \int_v^\infty \bar{G}\left(\frac{w - \lambda x}{\sqrt{1 - \lambda^2}}\right) dF(x) &= \int_{-\infty}^\infty \bar{G}\left(\frac{w - \lambda x}{\sqrt{1 - \lambda^2}}\right) dF(x) - \int_{-\infty}^v \bar{G}\left(\frac{w - \lambda x}{\sqrt{1 - \lambda^2}}\right) dF(x) \\ &\leq \int_{-\infty}^\infty \bar{G}\left(\frac{w - \lambda' x}{\sqrt{1 - \lambda'^2}}\right) dF(x) - \int_{-\infty}^v \bar{G}\left(\frac{w - \lambda' x}{\sqrt{1 - \lambda'^2}}\right) dF(x) \\ &= \int_v^\infty \bar{G}\left(\frac{w - \lambda' x}{\sqrt{1 - \lambda'^2}}\right) dF(x). \end{aligned} \quad (3.2.11)$$

From (3.2.7), (3.2.9) and (3.2.10), it follows that the family of random variables  $\{(W_\lambda, V), -1 < \lambda \leq 0\}$  is decreasingly NQD in  $\lambda$ . The family of bivariate normal distributions with negative correlation coefficient is a member of this family.

### 3.3. Closure Properties of $(\bar{\beta}, \succ_{\text{NQD}})$ .

In this section we establish preservation of the NQD ordering under combination, mixture, transformations of the random variable by increasing (decreasing) function, limit in distribution and other

operations of interest in statistics. First note that (3.1.1) is also equivalent to

$$\text{Cov}_{H_2} [f(X), g(Y)] \leq \text{Cov}_{H_1} [f(X), g(Y)]$$

for all increasing  $f$  and  $g$ .

Next we show that the ordering is preserved under combination.

Lemma 1. Let  $\tilde{X} = (X_1, X_2)$  and  $\tilde{Y} = (Y_1, Y_2)$  have distributions  $H_1$  and  $H_2$  respectively, where  $H_1$  and  $H_2$  belong to  $\bar{\beta}$  such that  $\tilde{X} \succ_{\text{NQD}} \tilde{Y}$ , and  $\tilde{Z} = (Z_1, Z_2)$  with an arbitrary NQD distribution  $H$  independent of both  $\tilde{X}$  and  $\tilde{Y}$ . Then  $\tilde{X} + \tilde{Z} \succ_{\text{NQD}} \tilde{Y} + \tilde{Z}$ .

Proof.  $\text{Cov}(f(X_1 + Z_1), g(X_2 + Z_2)) = \text{Cov}[E\{f(X_1 + Z_1)|Z\}, E\{g(X_2 + Z_2)|Z\}] + E_Z\{\text{Cov}(f(X_1 + Z_1), g(X_2 + Z_2))|Z\} \geq 0$ .

Note that the first and second terms are greater than or equal to zero for any increasing function  $f$  and  $g$ . So  $\tilde{X} + \tilde{Z}$  is NQD, similarly we can show that  $\tilde{Y} + \tilde{Z}$  is also NQD. Now for showing  $\tilde{X} + \tilde{Z} \succ_{\text{NQD}} \tilde{Y} + \tilde{Z}$  we have to show,

$$\text{Cov}(f(X_1 + Z_1), g(X_2 + Z_2)) \leq \text{Cov}(f(Y_1 + Z_1), g(Y_2 + Z_2))$$

i.e.,

$$E(f(X_1 + Z_1) g(X_2 + Z_2)) \leq E(f(Y_1 + Z_1) g(Y_2 + Z_2)) \quad (3.3.1)$$

for any increasing function  $f$  and  $g$ .

Now,

$$\begin{aligned}
 E(f(X_1 + Z_1) g(X_2 + Z_2)) &= E_Z (E(f(X_1 + Z_1) g(X_2 + Z_2)) | Z)) \\
 &= E_Z (E_X (f(X_1 + Z_1) g(X_2 + Z_2))) \leq E_Z (E_Y (f(Y_1 + Z_1) g(Y_2 + Z_2))) \\
 &= E(f(Y_1 + Z_1) g(Y_2 + Z_2)).
 \end{aligned}$$

The inequality comes from the fact that  $X \underset{\sim}{> \text{NQD}} Y$ .

Theorem 2. Suppose  $(X_i, Y_i)$  and  $(U_i, V_i)$  are such that  $(X_i, Y_i) \underset{\sim}{> \text{NQD}} (U_i, V_i)$  for  $i = 1, 2$ . Further, let  $(X_1, Y_1)$  and  $(X_2, Y_2)$  be independent and  $(U_1, V_1)$  and  $(U_2, V_2)$  be independent. Then  $(X_1 + X_2, Y_1 + Y_2)$  is more negative quadrant dependent than  $(U_1 + U_2, V_1 + V_2)$ .

Proof. Use Lemma 1 and the method which Ahmed et al. (1979) used proving their Theorem 4.2.

The next theorem deals with the preservation of the NQD ordering under mixtures. We may define the class  $\bar{\beta}_\lambda = \{H_\lambda : H(x, \infty) = F(x|\lambda), H(\infty, y) = G(y|\lambda), H_\lambda|\lambda \text{ is NQD}, X \text{ is SD in } \lambda \text{ and } Y \text{ is SI in } \lambda \text{ or } X \text{ is SI and } Y \text{ is SD in } \lambda\}$ .

Consider  $(\bar{\beta}_\lambda, \underset{\sim}{> \text{NQD}})$ . The following proposition shows that if two elements of  $\bar{\beta}_\lambda$  are ordered according to  $\underset{\sim}{> \text{NQD}}$ , then after mixing on  $\lambda$ , the resulting elements in  $\bar{\beta}$  preserve the same order.

Theorem 3. Let  $(X_1, X_2)|\lambda$  and  $(Y_1, Y_2)|\lambda$  belong to  $\bar{\beta}_\lambda$  and let

$(X_1, X_2) | \lambda \succ_{\text{NQD}} (Y_1, Y_2) | \lambda$  for all  $\lambda$ . Then, unconditionally,  $(X_1, X_2), (Y_1, Y_2)$  belongs to  $\bar{\beta}$  and  $(X_1, X_2) \succ_{\text{NQD}} (Y_1, Y_2)$ .

Proof. From Chapter 2  $(X_1, X_2)$  and  $(Y_1, Y_2)$  are NQD.

Now,

$$\begin{aligned} E(f(X_1) g(X_2)) &= E_{\lambda} (E(f(X_1) g(X_2)) | \lambda) \leq E_{\lambda} (E(f(Y_1) g(Y_2)) | \lambda) \\ &= E(f(Y_1) g(Y_2)). \end{aligned}$$

Next, we show the NQD ordering is invariant under transformation of univariate increasing (decreasing) function.

Theorem 4. Let  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  be increasing (decreasing) func-

tion, and let  $\beta_{f,g} = \{(f(X), g(Y)): (X, Y) \in \bar{\beta}\}$  be the class of transformations of the member  $\bar{\beta}$  under  $f, g$ . If two ordered elements belong to  $\bar{\beta}$ , then the corresponding elements in  $\bar{\beta}_{f,g}$  maintain the same order.

Theorem 5. Let  $\{(X_i, Y_i)\}_{i=1}^n$ ,  $i = 1, \dots, n$  be  $n$ -independent pairs from a bivariate distribution  $H_j$ ,  $j = 1, 2$ . Suppose  $H_1$  and  $H_2$  such that  $H_1 \succ_{\text{NQD}} H_2$ . Then for every pair  $(f, g)$  of concordant functions,

$$\begin{aligned} \text{Cov}_{H_1} [f(X_1, \dots, X_n), g(Y_1, \dots, Y_n)] &\leq \text{Cov}_{H_2} [f(X_1, \dots, X_n), \\ &\quad g(Y_1, \dots, Y_n)]. \end{aligned} \tag{3.3.2}$$

In the following theorem we show that the ordering is preserved



under limits in distributions.

**Theorem 6.** Suppose  $H_n \xrightarrow{\text{NQD}} H'_n$  for every  $n$ , and  $H_n, H'_n$  converge weakly to  $H, H'$ , respectively. Then

$$H \xrightarrow{\text{NQD}} H'$$

Proof. Use Theorem 4 in Chapter 2 and the proposition 4.2 in Ahmed et al. (1979).

### 3.4. Applications

In this section we consider a few sample applications to show the potential applicability of the theoretical results obtained earlier.

From Theorem 20 of Chapter 2, we know that Kendall's  $\tau$ , Spearman's  $\rho_s$  and the quadrant measure  $q$  are all negative under some circumstances. An immediate consequence of Theorem 5 is the following theorem.

Theorem 7. Let  $H_1$  and  $H_2$  be such that  $H_1 \xrightarrow{\text{NQD}} H_2$ . Then Kendall's  $\tau$ , Spearman's  $\rho_s$ , and  $q$  satisfy

$$\tau_{H_1} \leq \tau_{H_2}, \quad \rho_{s_{H_1}} \leq \rho_{s_{H_2}}, \quad \text{and} \quad q_{H_1} \leq q_{H_2}.$$

Another area of interesting and useful application of ordering NQD random variables is in bivariate Markov chains.

In many practically occurring Markov chains, it is frequently of interest to know the degree of change in the dependence from one

transitional step to the next, when the degree of dependence follows some pattern depending on the step number.

Consider the bivariate Markov sequence  $\{(X_0, Y_0), \dots, (X_n, Y_n)\}$  such that  $Z_0 = (X_0, Y_0)$  has the bivariate normal (BVN) distribution with  $\mu_0 = (0, 0)$  and  $\Sigma_{0,0} = \begin{pmatrix} 1 & \rho_0 \\ \rho_0 & 1 \end{pmatrix}$ ,  $-1 < \rho_0 < 0$ . Furthermore assume  $Z_{i+1} | Z_i = z_i$  has the BVN  $(\Sigma_{i,i+1} \Sigma_{i,i}^{-1} z_i, \Sigma_{i+1,i+1.i})$ , where

$$\Sigma_{i,i+1} = \begin{bmatrix} a_{i+1} & b_{i+1} \\ b_{i+1} & a_{i+1} \end{bmatrix}, \quad b_{i+1} = a_{i+1} \rho_i,$$

$$\Sigma_{i+1,i+1.i} = \Sigma_{i+1,i+1} - \Sigma_{i,i+1} \Sigma_{i,i}^{-1} \Sigma_{i+1,i}$$

$$= \begin{bmatrix} 1 - a_{i+1}^2 & 1 - \frac{a_{i+1}^2 (1+\rho_i)}{1 - \rho_i^2} \\ 1 - \frac{a_{i+1}^2 (1+\rho_i)}{1 - \rho_i^2} & 1 - a_{i+1}^2 \end{bmatrix} \quad \text{and}$$

$$1 > a_{i+1} > \frac{1 - \rho_i}{\sqrt{1 - \rho_i + \rho_i^2}}, \quad i = 0, 1, 2, \dots, n-1.$$

Clearly,  $X_i \sim N(0, 1)$  and  $Y_i \sim N(0, 1)$  for  $i = 0, 1, 2, \dots$ ; i.e., the sequence has fixed marginals. Furthermore, simple calculations yield

$$(X_i, Y_i) \succ_{\text{NQD}} (X_{i-1}, Y_{i-1}), \quad i = 1, \dots, n.$$

#### 4. MULTIVARIATE NEW BETTER THAN USED AND NEW BETTER THAN USED IN EXPECTATION

##### 4.1. Multivariate New Better Than Used

Let  $X_1, \dots, X_p$  denote the survival times of  $p$  devices having a joint distribution function  $H_p(x_1, \dots, x_p)$ . The joint survival function of these  $p$  devices is denoted by  $\bar{H}_p(x_1, \dots, x_p) = P(X_1 > x_1, \dots, X_p > x_p)$ . It is assumed that  $\bar{H}_p(0, 0, \dots, 0) = 1$ . Note that when  $p = 1$ ,  $H_1$  is said to satisfy the NBU (NWU) property if

$$\bar{H}_1(t_1 + x_1) \leq (\geq) \bar{H}_1(t_1) \bar{H}_1(x_1) \quad (4.1.1)$$

for all  $x_1 \geq 0, t_1 \geq 0$ .

This is equivalent to saying that the life length of a new unit is stochastically larger (smaller) than the life length of an unfailed unit at age  $t_1$  for  $t_1 \geq 0$ . Note that equality in (4.1.1) holds if and only if  $H_1$  is an exponential distribution.

We consider the following definitions of MNBU (MNWU) each of generalizes (4.1.1). Write  $\underline{x} = (x_1, \dots, x_p)$ ,  $\underline{t} = (t_1, \dots, t_p)$  and  $\underline{1} = (1, \dots, 1)$ . We say  $\underline{x} = (a_1, \dots, a_p) \geq \underline{0}$  if each  $a_i \geq 0$  ( $1 \leq i \leq p$ ).

Definitions.  $H_p$  is said to be

(i) MNBU-I (MNWU-I) if

$$\bar{H}_p(\underline{x} + \underline{t}) \leq (\geq) \bar{H}_p(\underline{x}) \bar{H}_p(\underline{t}) \quad (4.1.2)$$

for all  $\underline{x} \geq \underline{0}, \underline{t} \geq \underline{0}$ ;

(ii) MNBU-II (MNWU-II) if

$$\overline{H}_{\underset{\sim}{p}}(\underset{\sim}{x} + \underset{\sim}{t}) \leq (\geq) \overline{H}_{\underset{\sim}{p}}(\underset{\sim}{x}) \overline{H}_{\underset{\sim}{p}}(\underset{\sim}{t}) \quad (4.1.3)$$

for all similarly ordered  $\underset{\sim}{x}$  and  $\underset{\sim}{t}$  satisfying  $\underset{\sim}{x} \geq \underset{\sim}{0}$  and  $\underset{\sim}{t} \geq \underset{\sim}{0}$ . Two vectors  $\underset{\sim}{x}$  and  $\underset{\sim}{t}$  are said to be similarly ordered if  $(x_i - x_j)(t_i - t_j) \geq 0$  for all  $1 \leq i < j \leq p$ ;

(iii) MNBU-III (MNWU-III) if

$$\overline{H}_{\underset{\sim}{p}}(\underset{\sim}{x} + \underset{\sim}{t} \mathbf{1}) \leq (\geq) \overline{H}_{\underset{\sim}{p}}(\underset{\sim}{x}) \overline{H}_{\underset{\sim}{p}}(\underset{\sim}{t} \mathbf{1}) \quad (4.1.4)$$

for all  $\underset{\sim}{x} \geq \underset{\sim}{0}, \underset{\sim}{t} \geq \underset{\sim}{0}$ , and similar inequalities hold for all marginal survival functions;

(iv) MNBU-IV (MNWU-IV) if

$$\overline{H}_{\underset{\sim}{p}}(\underset{\sim}{x} \mathbf{1} + \underset{\sim}{t} \mathbf{1}) \leq (\geq) \overline{H}_{\underset{\sim}{p}}(\underset{\sim}{x} \mathbf{1}) \overline{H}_{\underset{\sim}{p}}(\underset{\sim}{t} \mathbf{1}) \quad (4.1.5)$$

for all  $\underset{\sim}{x} \geq \underset{\sim}{0}, \underset{\sim}{t} \geq \underset{\sim}{0}$ , and similar inequalities hold for all marginal survival functions.

Remark 1. Definitions (i), (iii) and (iv) appear in Buchanan and Singpurwalla (1977), while definition (ii) appears in Marshall and Shaked (1979).

Remark 2. In definition (i), similar inequalities hold for all marginal survival functions by putting the appropriate  $x_i$ 's and corresponding  $t_i$ 's to be equal to zeros. A similar remark applies to definition (ii) since  $\underset{\sim}{x}$  and  $\underset{\sim}{t}$  are similarly ordered.

Remark 3. The definition (i) says that the conditional survival prob-

ability  $\frac{\bar{F}(\tilde{x} + \tilde{t})}{\bar{F}(\tilde{t})}$  of a unit with component attaining the respective ages  $t_1, \dots, t_p$  is less (greater) than or equal to the corresponding survival probability of a new unit. A similar interpretation can be given to the definition (ii) except that here  $\tilde{x}$  and  $\tilde{t}$  are also supposed to be similarly ordered. Definition (iii) can be given a two-fold interpretation. On the one hand it says that the conditional survival probability for a unit with each component attaining the age  $t$  is less (greater) than or equal to the corresponding survival probability for a new unit. Alternatively, this can be interpreted as the conditional survival probability for each component of the unit to survive an additional period of time  $t$  given that the components have reached the respective ages  $x_1, \dots, x_p$  is less (greater) than or equal to the corresponding survival probability for a new unit. Definition (iv) says that a random vector is MNBU (MNWU) if the minimum of each subset has a NBU (NWU) distribution.

Remark 4. It is immediate to see that

$$\text{MNBU-I} \Rightarrow \text{MNBU-II} \Rightarrow \text{MNBU-III} \Rightarrow \text{MNBU-IV}; \quad (4.1.6)$$

$$\text{MNWU-I} \Rightarrow \text{MNWU-II} \Rightarrow \text{MNWU-III} \Rightarrow \text{MNWU-IV}. \quad (4.1.7)$$

No converse implication holds for either (4.1.6) and (4.1.7). The following three examples justify this assertion for (4.1.6). Similar

examples maybe constructed to illustrate (4.1.7).

Example 1. (MNBU-II  $\nrightarrow$  MNBU-I). Let  $X_1$  and  $X_2$  have the bi-variate Marshall-Olkin (1967) exponential distribution with survival function

$$\bar{H}_2(x_1, x_2) = \exp[-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max(x_1, x_2)], \quad (4.1.8)$$

$x_1 \geq 0, x_2 \geq 0, \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_{12} > 0$ . Then, for  $x_1 \geq 0, x_2 \geq 0$ ,

$t_1 \geq 0, t_2 \geq 0$ ,

$$\frac{\bar{H}_2(x_1 + t_1, x_2 + t_2)}{[\bar{H}_2(x_1, x_2) \bar{H}_2(t_1, t_2)]} = \exp[-\lambda_{12} \{\max(x_1 + t_1, x_2 + t_2) - \max(x_1, x_2) - \max(t_1, t_2)\}]. \quad (4.1.9)$$

The right hand side of (4.1.9) equals 1 when  $(x_1 - x_2)(t_1 - t_2) \geq 0$ , so that the above family of distributions is MNBU-II. However,  $\bar{H}_2(7+3, 6+5) > \bar{H}_2(7, 6) \bar{H}_2(3, 5)$  so that no member in the family of distributions given in (4.1.8) is MNBU-I.

Example 2. (MNBU-III  $\nrightarrow$  MNBU-II). Let  $X_1$  and  $X_2$  have the joint survival function

$$\bar{H}_2(x_1, x_2) = \begin{cases} \frac{1}{2} [\exp(-x_1) + \exp(-x_2)], & 0 \leq x_1, x_2 < 5 \\ \exp[-\max(x_1, x_2)] & , \text{ otherwise.} \end{cases} \quad (4.1.10)$$

Then,  $\bar{H}_2(t, t) = \exp(-t)$  for all  $t \geq 0$  and  $\bar{H}_2(x_1 + t, x_2 + t) \leq \bar{H}_2(x_1, x_2) \bar{H}_2(t, t)$  for all  $x_1 \geq 0, x_2 \geq 0$  and  $t \geq 0$ ; also the marginal

distribution of  $X_1$  is given by

$$\begin{aligned}\overline{H}_1(x_1) &= \frac{1}{2} [1 + \exp(-x_1)], \quad 0 \leq x_1 < 5 \\ &= \exp(-x_1) \quad \text{otherwise}\end{aligned}\tag{4.1.11}$$

is marginally NBU. Similarly  $X_2$  is marginally NBU. Thus  $H_2$

is MNBU-III. However, for  $(x_1 - x_2)(t_1 - t_2) > 0$  such that

$0 < x_1 + t_1 < 5$ ,  $0 < x_2 < 5$ , one has

$$\begin{aligned}\overline{H}_2(x_1 + t_1, x_2 + t_2) - \overline{H}_2(x_1, x_2)\overline{H}_2(t_1, t_2) \\ &= \frac{1}{2} [\exp(-x_1 - t_1) + \exp(-x_2 - t_2)] \\ &\quad - \frac{1}{4} [\exp(-x_1) + \exp(-x_2)] [\exp(-t_1) + \exp(-t_2)] \\ &= \frac{1}{4} [\exp(-x_1 - t_1) + \exp(-x_2 - t_2) - \exp(-x_1 - t_2) - \exp(-x_2 - t_1)] \\ &= \frac{1}{4} [\exp(-x_1) - \exp(-x_2)] [\exp(-t_1) - \exp(-t_2)] > 0,\end{aligned}$$

so that the distribution is not MNBU-II.

Example 3. (MNBU-IV  $\nrightarrow$  MNBU-III). Let  $X_1$  and  $X_2$  have the joint survival function

$$\overline{H}_2(x_1, x_2) = \exp[-\lambda_1 x_1 - \lambda_2 x_2 - \max(\lambda_3 x_1, \lambda_4 x_2)], \tag{4.1.12}$$

where  $\lambda_1 + \lambda_3 > 0$ ,  $\lambda_2 + \lambda_4 > 0$ ,  $\lambda_i \geq 0$  ( $1 \leq i \leq 4$ ). This bivariate distribution appears in Marshall and Olkin (1967) and in Esary and Marshall (1974). Then each  $X_1$  and  $X_2$  has a marginal exponential distribution, and  $\min(X_1, X_2)$  has also an exponential distribution.



However, since,

$$\frac{\overline{H}_2(x_1 + t, x_2 + t)}{\overline{H}_2(x_1, x_2) \overline{H}_2(t, t)} = \exp [\max(\lambda_3 x_1, \lambda_4 x_2) + t \max(\lambda_3, \lambda_4) - \max(\lambda_3 x_1 + \lambda_3 t, \lambda_4 x_2 + \lambda_4 t)]$$

if  $\lambda_3 < \lambda_4, \lambda_3 x_1 > \lambda_4 x_2$  and  $\lambda_3 x_1 + \lambda_3 t > \lambda_4 x_2 + \lambda_4 t$  (for example take  $\lambda_3 = 2, \lambda_4 = 3, x_1 = 2, x_2 = 1, t = \frac{1}{2}$ ), then  $\overline{H}_2(x_1 + t, x_2 + t) > \overline{H}_2(x_1, x_2) \overline{H}_2(t, t)$  so that the distribution is not MNBU-III.

Next we consider several important classes of life distribution, satisfying one or the other of the MNBU definitions. For simplicity, we restrict ourselves only to the bivariate case.

Example 4. Consider the bivariate Gumbel family of distributions (see Gumbel (1961) or Johnson and Kotz (1972, p. 261)) with joint survival function

$$\overline{H}_2(x_1, x_2) = \exp(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_1 x_2), \quad (4.1.13)$$

$x_1 \geq 0, x_2 \geq 0, \lambda_i > 0$  ( $i = 1, 2, 3$ ). Then, for  $x_i \geq 0, t_i \geq 0$  ( $i = 1, 2$ ),

$$\frac{\overline{H}_2(x_1 + t_1, x_2 + t_2)}{\overline{H}_2(x_1, x_2) \overline{H}_2(t_1, t_2)} = \exp[-\lambda_3(x_1 t_2 + x_2 t_1)] < 1,$$

so that this family of distributions is MNBU-I.

Example 5. Consider the bivariate distribution with joint survival function

$$\overline{H}_2(x_1, x_2) = 1 - x_1 x_2, \quad 0 \leq x_1, x_2 \leq 1. \quad (4.1.14)$$

Then, for  $x_i \geq 0, t_i \geq 0, 0 \leq x_1 + t_1, x_2 + t_2 \leq 1$ , one has

$$\begin{aligned} \overline{H}_2(x_1, x_2) \overline{H}_2(t_1, t_2) - \overline{H}_2(x_1 + t_1, x_2 + t_2) &= (1 - x_1 x_2)(1 - t_1 t_2) - \\ [1 - (x_1 + t_1)(x_2 + t_2)] &= x_1 x_2 t_1 t_2 + x_1 t_2 + x_2 t_1 \geq 0, \end{aligned}$$

so that the distribution is MNBU-I.

Example 6. Let  $Y_1, Y_2$  and  $Y_3$  be independently distributed random variables,  $Y_i$  having the distribution function  $F_i$ , each  $F_i$  being a NBU distribution satisfying  $F_i(z) = 0$  for  $z < 0$  ( $1 \leq i \leq 3$ ). Let  $X_1 = \min(Y_1, Y_3)$ ,  $X_2 = \min(Y_2, Y_3)$ . Then  $X_1$  and  $X_2$  have the joint survival function

$$\overline{H}_2(x_1, x_2) = \overline{F}_1(x_1) \overline{F}_2(x_2) \overline{F}_3(\max(x_1, x_2)). \quad (4.1.15)$$

Then for  $x_i \geq 0, t_i \geq 0$  ( $i = 1, 2$ ) and  $(x_1 - x_2)(t_1 - t_2) \geq 0$ ,

$$\overline{H}_2(t_1 + x_1, t_2 + x_2) \leq \overline{H}_2(x_1, x_2) \overline{H}_2(t_1, t_2),$$

so that the bivariate distribution is MNBU-II. The bivariate Marshall-Olkin (1967) family of distributions belong to this class.

Another subclass is the bivariate weibull family of distributions

$$\overline{H}_2(x_1, x_2) = \exp [-\lambda_1 x_1^{a_1} - \lambda_2 x_2^{a_2} - \lambda_3 (\max(x_1, x_2))^{a_3}],$$

$x_i > 0, a_i \geq 1$  ( $1 \leq i \leq 3$ ). This family of distributions is mentioned in David (1974) and in Lee and Thompson (1974). As noted already in Example 1 in the special case of the Marshall-Olkin family of bivariate exponential distributions, this family of distribution is not MNBU-I.

Example 7. Consider the following subclass of the Freund family of bivariate distributions (see Freund (1961) or Johnson and Kotz (1972, pp. 263-264)) with joint survival function

$$\begin{aligned}
 \overline{H}_2(x_1, x_2) &= \beta(x_1 - x_2) \exp[-(\alpha + \beta)x_1] + \exp[-(\alpha + \beta)x_2], \\
 0 &\leq x_2 \leq x_1 \\
 &= \alpha(x_2 - x_1) \exp[-(\alpha + \beta)x_2] + \exp[-(\alpha + \beta)x_1], \\
 0 &\leq x_1 \leq x_2,
 \end{aligned} \tag{4.1.16}$$

with  $\alpha > 0$ ,  $\beta > 0$ . Now for  $x_1 > x_2$ ,  $t_1 > t_2$ ,

$$\begin{aligned}
 &\overline{H}_2(x_1, x_2) \overline{H}_2(t_1, t_2) - \overline{H}_2(x_1 + t_1, x_2 + t_2) \\
 &= \beta^2(x_1 - x_2)(t_1 - t_2) \exp[-(\alpha + \beta)(x_1 + t_1)] \\
 &+ \beta(t_1 - t_2) \exp[-(\alpha + \beta)(x_2 + t_1)] \\
 &+ \beta(x_1 - x_2) \exp[-(\alpha + \beta)(x_1 + t_2)] \\
 &- \beta(x_1 + t_1 - x_2 - t_2) \exp[-(\alpha + \beta)(x_1 + t_1)] > 0.
 \end{aligned}$$

Similarly, for  $x_2 > x_1$ ,  $t_2 > t_1$ ,

$$\overline{H}_2(x_1, x_2) \overline{H}_2(t_1, t_2) - \overline{H}_2(x_1 + t_1, x_2 + t_2) > 0.$$

Hence, this family of distributions is MNBU-II.

Example 8. Esary and Marshall (1974) have investigated a very wide class of multivariate distributions with exponential minimums. Example 3 provides an illustration of this. Not all such distributions are

MNBU-IV as the marginal distributions need not be exponential.

To see this, consider the following example which also appears in

Esary and Marshall (1974).

$$\begin{aligned} \overline{F}(x_1, x_2) = & p \exp [-\xi_1 x_1 - \xi_2 x_2 - \xi_3 \max(x_1, x_2)] \\ & + (1-p) \exp [-\eta_1 x_1 - \eta_2 x_2 - \eta_3 \max(x_1, x_2)], \end{aligned} \quad (4.1.17)$$

where  $0 < p < 1$ ,  $\xi_i > 0$ ,  $\eta_i > 0$  ( $1 \leq i \leq 3$ ) and  $\sum_{i=1}^3 \xi_i = \sum_{i=1}^3 \eta_i$ .

Then  $\min(X_1, X_2)$  has an exponential distribution but neither  $X_1$  nor  $X_2$  marginally has an exponential distribution so that the joint distribution is not MNBU-IV.

Example 9. The bivariate Marshall-Olkin family of exponential distribution is MNWU-I since for all  $x_i \geq 0$ ,  $t_i \geq 0$  ( $i = 1, 2$ ),

$$\frac{\overline{H}_2(x_1 + t_1, x_2 + t_2)}{\overline{H}_2(x_1, x_2) \overline{H}_2(t_1, t_2)} \geq 1.$$

Remark 5. In view of Example 1 and Example 9, it follows that unlike the univariate case, the bivariate Marshall-Olkin family of distributions does not remain on the boundary of MNBU-I and MNWU-I classes of distributions. However, if one accepts any one of the other three definitions of MNBU and the corresponding one for MNWU, the Marshall-Olkin family of distributions remains on the boundary of these two classes.

Example 10. Let  $X_1, X_2$  have the joint survival function

$$\overline{H}_2(x_1, x_2) = (1 + \lambda_1 x_1 + \lambda_2 x_2)^{-\alpha}, \quad (4.1.18)$$

for  $x_1 \geq 0$ ,  $x_2 \geq 0$ ,  $\alpha \geq 0$ ,  $\lambda_1 > 0$  and  $\lambda_2 > 0$ . Then,

$$\frac{\overline{H}_2(x_1 + t_1, x_2 + t_2)}{\overline{H}_2(x_1, x_2) \overline{H}_2(t_1, t_2)} \geq 1.$$

This family of bivariate distributions is MNWU-I.

Example 11. (MNWU-II  $\not\supset$  MNWU-I). Let

$$\overline{H}_2(x_1, x_2) = \exp \left[ -\lambda (\max(x_1, x_2))^{\frac{1}{2}} \right], \quad (4.1.19)$$

$\lambda > 0$ . Then for  $x_i \geq 0$ ,  $t_i \geq 0$

$$\frac{\overline{H}_2(x_1 + t_1, x_2 + t_2)}{\overline{H}_2(x_1, x_2) \overline{H}_2(t_1, t_2)} = \exp \left[ \lambda \left\{ (\max(x_1 + t_1, x_2 + t_2))^{\frac{1}{2}} - \max(x_1, x_2)^{\frac{1}{2}} - \max(t_1, t_2)^{\frac{1}{2}} \right\} \right]$$

if  $(x_1 - x_2)(t_1 - t_2) \geq 0$ , then

$$\frac{\overline{H}_2(x_1 + t_1, x_2 + t_2)}{\overline{H}_2(x_1, x_2) \overline{H}_2(t_1, t_2)} \geq 1,$$

while

$$\overline{H}_2(3+5, 4+2) < \overline{H}_2(3, 4) \overline{H}_2(5, 2).$$

#### 4.2. Closure Properties of MNBU

In this section, we prove certain closure properties of MNBU distributions. Comparable closure properties for multivariate IFRA

distributions are given in Block and Savits (1977). Let  $A_1, A_2, A_3$  and  $A_4$  denote the class of life distributions satisfying the definitions (i), (ii), (iii) and (iv) respectively of MNBU. Then we have the following theorem.

Theorem 1.

- (P1)  $A_j$  ( $1 \leq j \leq 4$ ) is closed under limit in distribution.
- (P2)  $A_j$  ( $1 \leq j \leq 4$ ) is closed under formation of coherent systems.
- (P3) If  $(T_1, \dots, T_m) \in A_j$ , any subset of  $(T_1, \dots, T_m) \in A_j$  ( $1 \leq j \leq 4$ ).
- (P4) If  $(T_1, \dots, T_m) \in A_j$ ,  $(T_1, \dots, T'_n) \in A_j$  and  $(T_1, \dots, T_m)$  and  $(T'_1, \dots, T'_n)$  are independently distributed, then  $(T_1, \dots, T_m, T'_1, \dots, T'_n) \in A_j$  ( $1 \leq j \leq 4$ ).
- (P5) If  $(T_1, \dots, T_m) \in A_1$ , then  $(C_1 T_1, \dots, C_m T_m) \in A_1$  when  $C_i \geq 0$  ( $1 \leq i \leq m$ ) similarly, if  $(T_1, \dots, T_m) \in A_j$  ( $2 \leq j \leq 4$ ), then  $(CT_1, \dots, CT_m) \in A_j$  ( $2 \leq j \leq 4$ ) when  $C > 0$ .
- (P6)  $A_j$  is closed under convolution ( $1 \leq j \leq 4$ ) (whenever the operation is meaningful).

Proof.

(P1) Suppose for every  $k$ ,  $(T_{1k}, \dots, T_{mk}) \in A_j$  and  $(T_{1k}, \dots, T_{mk})$  converges weakly to  $(T_1, \dots, T_m)$  as  $k \rightarrow \infty$ . Then, using the appropriate definition of MNBU for  $(T_{1k}, \dots, T_{mk})$ , and taking limits, the result follows.

(P2) The result is an immediate consequence of Theorem 5.1 of Barlow and Proschan (1975, p. 182) noting that under any of the four definitions each component variable is marginally NBU.

(P3) The proofs come immediately from the definitions.

(P4) Suppose  $(T_1, \dots, T_m) \in A_1$  and  $(T'_1, \dots, T'_n) \in A_1$ . Let  $\bar{H}_{m+n}(t_1, \dots, t_m, t'_1, \dots, t'_n)$  denote the joint survival function of  $(T_1, \dots, T_m, T'_1, \dots, T'_n)$ , and let  $\bar{F}_m(t_1, \dots, t_m)$  and  $\bar{G}_n(t'_1, \dots, t'_n)$  denote the respective joint survival functions of  $(T_1, \dots, T_m)$  and  $(T'_1, \dots, T'_n)$ . Then for  $x_i \geq 0$  ( $1 \leq i \leq m$ ),  $x'_i \geq 0$  ( $1 \leq i \leq n$ ),  $t_i \geq 0$  ( $1 \leq i \leq m$ ) and  $t'_i \geq 0$  ( $1 \leq i \leq n$ ) one gets using the independence of  $(T_1, \dots, T_m)$  and  $(T'_1, \dots, T'_n)$  and the MNBU-I properties,

$$\begin{aligned} & \bar{H}_{m+n}(x_1 + t_1, \dots, x_m + t_m, x'_1 + t'_1, \dots, x'_n + t'_n) \\ &= \bar{F}_m(x_1 + t_1, \dots, x_m + t_m) \bar{G}_n(x'_1 + t'_1, \dots, x'_n + t'_n) \\ &\leq \bar{F}_m(x_1, \dots, x_m) \bar{F}_m(t_1, \dots, t_m) \bar{G}_n(x'_1, \dots, x'_n) \\ &\quad \bar{G}_n(t'_1, \dots, t'_n) \\ &= \bar{H}_{m+n}(x_1, \dots, x_m, x'_1, \dots, x'_n) \bar{H}_{m+n}(t_1, \dots, t_m, \\ &\quad t'_1, \dots, t'_n). \end{aligned}$$

Hence, the closure properties for  $A_1$  follows. Similarly, closure properties for  $A_2, A_3$  and  $A_4$  are proved.

(P5) Let  $\bar{H}_m(t_1, \dots, t_m)$  denote the joint survival function of

$(T_1, \dots, T_m)$  satisfying the MNBU-I property. Then for  $C_i > 0$ ,  $x_i \geq 0$ ,  $t_i \geq 0$  ( $1 \leq i \leq m$ ),

$$\begin{aligned}
 & P(C_1 T_1 > x_1 + t_1, \dots, C_m T_m > x_m + t_m) \\
 &= \overline{H}_m\left(\frac{(x_1 + t_1)}{C_1}, \dots, \frac{(x_m + t_m)}{C_m}\right) \\
 &\leq \overline{H}_m\left(\frac{x_1}{C_1}, \dots, \frac{x_m}{C_m}\right) \overline{H}_m\left(\frac{t_1}{C_1}, \dots, \frac{t_m}{C_m}\right) \\
 &= P(C_1 T_1 > x_1, \dots, C_m T_m > x_m) \\
 &\quad P(C_1 T_1 > t_1, \dots, C_m T_m > t_m).
 \end{aligned}$$

Hence,  $(C_1 T_1, \dots, C_m T_m)$  has a MNBU-I distribution. This proves the result for  $A_1$ . Similar proofs work for  $A_2, A_3$  and  $A_4$ .

(P6) To prove (P6) for  $A_1$ , we first obtain the following multivariate generalization of Lemma (3.4 (i)) of Block and Savits (1978).

Lemma 1.  $H_p$  is MNBU-I if and only if

$$\begin{aligned}
 & \int_{x_p}^{\infty} \dots \int_{x_1}^{\infty} u(z_1 - x_1, \dots, z_p - x_p) dH_p(z) \\
 & \leq \overline{H}_p(x) \int_0^{\infty} \dots \int_0^{\infty} u(z) dH_p(z)
 \end{aligned} \tag{4.2.1}$$

for every nonnegative nondecreasing real valued function  $u$  defined on  $(0, \infty)^p$ .

Proof of Lemma 1. The sufficiency part follows by taking



$$u(z) = \prod_{i=1}^p I_{(t_i, \infty)}(z_i),$$

where  $I$  is the usual indicator function. To prove necessity, first note that the integral inequality (4.2.1) is valid for all functions of the form  $u(z) = I_A(z)$  with  $A = (t_1, \infty) \times \dots \times (t_p, \infty)$  or  $[t_1, \infty) \times \dots \times [t_p, \infty)$ ,  $t_i \geq 0$  ( $1 \leq i \leq p$ ). A nondecreasing nonnegative function can be written as a nondecreasing limit of nonnegative linear combinations of such  $u(z)$ , and hence for such functions (4.2.1) holds.

To use the lemma in proving the convolution property in the MNBU-I case let  $\tilde{X} = (X_1, \dots, X_p)$  and  $\tilde{Y} = (Y_1, \dots, Y_p)$  be two independent random vectors each having a MNBU-I distribution. Let  $\tilde{W} = \tilde{X} + \tilde{Y}$ . Let  $F_p, G_p$  and  $H_p$  denote the respective joint distributions of  $\tilde{X}, \tilde{Y}$  and  $\tilde{W}$ . We need to show that

$$\begin{aligned} & \int_{w_p}^{\infty} \dots \int_{w_1}^{\infty} u(z_1 - w_1, \dots, z_p - w_p) dH_p(z) \\ & \leq \overline{H}_p(w) \int_0^{\infty} \dots \int_0^{\infty} u(z) dH_p(z), \end{aligned} \quad (4.2.2)$$

for every nonnegative nondecreasing real  $u$  on  $(0, \infty)^p$ . Note that, the left hand side of (4.2.2)

$$= E \left[ u(W_1 - w_1, \dots, W_p - w_p) \prod_{i=1}^p I_{[W_i > w_i]} \right]$$

$$\begin{aligned}
&= E \left[ u(X_1 + Y_1 - w_1, \dots, X_p + Y_p - w_p) \prod_{i=1}^p I_{[X_i + Y_i > w_i]} \right] \\
&= E \left[ u(X_1 + Y_1 - w_1, \dots, X_p - Y_p - w_p) \prod_{i=1}^p I_{[Y_i > w_i]} \right] \\
&\quad + u(X_1 + Y_1 - w_1, \dots, X_p + Y_p - w_p) \prod_{i=1}^p I_{[Y_i \leq w_i, X_i + Y_i > w_i]} \\
&= E(B_1) + E(B_2) \text{ (say).} \tag{4.2.3}
\end{aligned}$$

Note that using the MNBU-I property of  $\tilde{X}$  the independence of  $\tilde{X}$  and  $\tilde{Y}$  and Lemma 1,

$$\begin{aligned}
&E(B_2 | Y_i = y_i (1 \leq i \leq p), y_i \leq w_i) \\
&\leq \bar{F}_p(w_1 - y_1, \dots, w_p - y_p) E(u(X_1, \dots, X_p)) \\
&\leq \bar{F}_p(w_1 - y_1, \dots, w_p - y_p) E(u(W_1, \dots, W_p)),
\end{aligned}$$

since  $\overset{\text{St}}{W_i} \geq X_i$  ( $1 \leq i \leq p$ ), and  $u$  is nondecreasing in its arguments.

Hence,

$$\begin{aligned}
E(B_2) &\leq \left[ \int_0^{w_p} \dots \int_0^{w_1} \bar{F}_p(w_1 - y_1, \dots, w_p - y_p) dG(y_1, \dots, y_p) \right] \\
&\quad E(u(W_1, \dots, W_p)). \tag{4.2.4}
\end{aligned}$$

Again, keeping  $X_i$  fixed at  $x_i$  ( $1 \leq i \leq p$ ) and writing  $u_0(y_1, \dots, y_p)$   $u(x_1 + y_1, \dots, x_p + y_p)$ , one gets after using the independence of  $\tilde{X}$  and  $\tilde{Y}$ , the MNBU-I property of  $\tilde{Y}$  and Lemma 1,

$$\begin{aligned}
E(B_1 | X_i = x_i \ (1 \leq i \leq p)) &= E[u_0(Y_1 - w_1, \dots, Y_p - w_p) \\
&\quad \prod_{i=1}^p I_{[Y_i > y_i]}] \\
&\leq \bar{G}_p(w_1, \dots, w_p) E u_0(Y_1, \dots, Y_p) \\
&= \bar{G}_p(w_1, \dots, w_p) E u(x_1 + Y_1, \dots, x_p + Y_p)
\end{aligned}$$

Hence,

$$E(B_1) \leq \bar{G}_p(w_1, \dots, w_p) E[u(W_1, \dots, W_p)] \quad (4.2.5)$$

from (4.2.3) - (4.2.5),

left hand side of (4.2.2)

$$\begin{aligned}
&\leq \left[ \int_0^{w_p} \dots \int_0^{w_1} \bar{F}_p(w_1 - y_1, \dots, w_p - y_p) dG(y_1, \dots, y_p) \right. \\
&\quad \left. + \bar{G}_p(w_1, \dots, w_p) \right] E u(W_1, \dots, W_p) \\
&= P(X_1 + Y_1 > w_1, \dots, X_p + Y_p > w_p) E u(W_1, \dots, W_p) \\
&= \bar{H}_p(w_1, \dots, w_p) E u(W_1, \dots, W_p) = \text{right hand side of (4.2.2)}.
\end{aligned}$$

This proves the result for MNBU-I. Similar proofs can be given for MNBU-II, III and IV.

Remark 6. Properties (P1) - (P5) hold for MNWU distribution.

However, even in the one dimensional case NWU distributions are not closed under convolutions (see Barlow and Proschan (1975,

p. 185)). Hence, property (P6) does not hold for MNWU distributions under any of the four definitions.

Remark 7. It is immediate from the definitions that the minimum over any subset of a MNBU (MNWU) random vector has itself a NBU (NWU) distribution.

#### 4.3. Relationship Between MIFRA and MNBU

In the univariate case, a life distribution function  $H_1$  is said to satisfy the IFRA property if  $\frac{-\log \bar{H}_1(x)}{x}$  is nondecreasing in  $x \geq 0$ . Bryson and Siddiqui (1969), and Marshall and Proschan (1972) have shown in the univariate case that  $\text{IFRA} \supset \text{NBU}$ , and that the converse implication does not hold. It is of interest to know whether a similar implication holds in the multidimensional case.

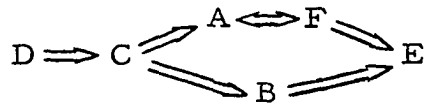
Esary and Marshall (1979) have given several definitions of the MIFRA based on multivariate generalizations of various characterizations of the univariate IFRA distributions. Yet another definition of the MIFRA based on a multivariate generalization of another characterization of the univariate IFRA is given in Block and Savits (1977).

We first introduce the following definitions of MIFRA given in Esary and Marshall (1979).

A  $p$  variate distribution function  $H_p$  is said to be MIFRA if

- A.  $-\log H_p(ax_1, \dots, ax_p)/a$  is nondecreasing in  $a > 0$  whenever  $\underline{x} \geq \underline{0}$ .
- B. For all coherent life functions  $\tau$ ,  $\tau(X_1, \dots, X_p)$  has an IFRA distribution.
- C.  $(X_1, \dots, X_p)$  has the representation  $X_i = \tau_i(Y_1, \dots, Y_k)$  ( $1 \leq i \leq p$ ) for some independent IFRA random variables  $Y_1, \dots, Y_k$  and some coherent life function  $\tau_1, \dots, \tau_p$  of order  $k$ .
- D. For some independent IFRA random variables  $Y_1, \dots, Y_k$  and nonempty subsets  $B_i$  of  $\{1, 2, \dots, k\}$ ,  $X_i = \min_{\ell \in B_i} Y_\ell$  ( $1 \leq i \leq p$ ).
- E. For all nonempty subsets  $B$  of  $\{1, 2, \dots, p\}$ ,  $\min_{i \in B} X_i$  is IFRA.
- F.  $\min_{1 \leq i \leq p} a_i X_i$  is IFRA whenever  $\underline{a} = (a_1, \dots, a_p) \geq \underline{0}$ .

Esary and Marshall (1979) have shown that the following and only the following implications hold.



We first explore the interrelationship of the A-F definitions of MIFRA with MNBU-I to IV.

Example 12.  $(D \not\Rightarrow \text{MNBU-I})$ . Consider once again the bivariate

Marshall-Olkin family of distributions defined in (4.1.8). Then

$$(X_1, X_2) \stackrel{d}{=} (\min(Y_1, Y_3), \min(Y_2, Y_3)), \text{ where } Y_1, Y_2 \text{ and } Y_3 \text{ are}$$

independent exponential random variables with respective parameters

$\lambda_1, \lambda_2$  and  $\lambda_{12}$ . Hence the joint distribution of  $X_1$  and  $X_2$  is MIFRA according to D, but (as shown already) it is not MNBU-I.

Remark 8. It follows as a consequence of Example 1 that neither of the other definitions of MIFRA  $\Rightarrow$  MNBU-I.

Next we show that  $D \Rightarrow$  MNBU-II. If  $(X_1, \dots, X_p)$  has a joint distribution belonging to D,  $\bar{H}_p(x_1, \dots, x_p)$  takes the form

$$\bar{H}_p(x_1, \dots, x_p) = \bar{F}_1\left(\max_{i \ni i \in S_p} x_i\right) \dots \bar{F}_k\left(\max_{i \ni i \in S_k} x_i\right),$$

where  $S_j = \{B_i : B_i \subset (1, 2, \dots, k), B_i \text{ contains } j\}$ ,  $1 \leq j \leq k$ .

If some  $S_j$  is empty, we define  $\max_{i \ni i \in S_j} x_i = 0$ . Also from the

definition each  $F_j$  is IFRA hence NBU ( $1 \leq j \leq p$ ). Thus for

similarly ordered  $\underline{x}$  and  $\underline{t}$  ( $x \geq 0, t \geq 0$ ),

$$\bar{H}_p(\underline{x} + \underline{t}) \leq \bar{H}_p(\underline{x}) \bar{H}_p(\underline{t}).$$

Next we provide an example to show that  $C \not\Rightarrow$  MNBU-III (so that neither A, F, B or E  $\Rightarrow$  MNBU-III).

To see this consider once again the Example 3. In this case

$(X_1, X_2) \stackrel{d}{=} (\frac{U_1}{b_1}, \frac{U_2}{b_2})$  for some  $b_1, b_2 > 0$ , where  $(U_1, U_2)$  has the

distribution given in (4.1.8). Since  $(U_1, U_2) \stackrel{d}{=} (\min(Y_1, Y_3)$

$\min(Y_2, Y_3))$ , where  $Y_1, Y_2, Y_3$  are independent exponential

variables with respective parameters  $\lambda_1, \lambda_2$  and  $\lambda_{12}$ , one has

$(X_1, X_2) \stackrel{d}{=} (\min(\frac{Y_1}{b_1}, \frac{Y_3}{b_1}), \min(\frac{Y_2}{b_2}, \frac{Y_3}{b_3}))$ . Thus  $(X_1, X_2) \stackrel{d}{=} (\tau_1(Y_1, Y_2, Y_3), \tau_2(Y_1, Y_2, Y_3))$  for some coherent life function  $\tau_1$  and  $\tau_2$  where  $Y_1, Y_2$  and  $Y_3$  are independent exponential and hence IFRA random variables. Hence, the joint distribution is MIFRA according to C. However, we have seen already that this distribution is not MNBU-III. Thus neither C nor any of A, F, B and E implies MNBU-II.

The final conclusion, therefore, is

- (i) none of A-F implies MNBU-I;
- (ii) only D implies both MNBU-II and MNBU-III;
- (iii) all of A-F imply MNBU-IV.

We conclude this section with a discussion of the MIFRA definition as given in Block and Savits (1977). According to these authors  $(X_1, \dots, X_p)$  has a MIFRA distribution if

$$Eh[X_1, \dots, X_p] \leq E^{\frac{1}{a}}[h^a(\frac{X_1}{a}, \dots, \frac{X_p}{a})], \quad (4.3.1)$$

for all nonnegative nondecreasing  $h$  and all  $0 < a \leq 1$ . This definition generalizes a corresponding univariate characterization of IFRA distributions by Block and Savits (1976). If  $(X_1, \dots, X_p)$  satisfies (4.3.1), then it follows from property P1 of Block and Savits (1977) that  $Y = \min_{i \in B} X_i$  where  $B \subset \{1, 2, \dots, p\}$  satisfies

$$E(h(Y)) \leq E^{\frac{1}{a}} h^a(Y), \quad (4.3.2)$$

for all nonnegative nondecreasing  $h$  and all  $0 < a \leq 1$ . Hence, from Block and Savits (1976),  $Y$  has an IFRA and hence NBU distribution. Thus, definition (4.3.1) of MIFRA implies MNBU-IV. However, Example 3 again illustrates that this definition does not imply MNBU-III. To see this recall if  $(X_1, X_2)$  has the distribution given in Example 3,  $(X_1, X_2) \stackrel{d}{=} (\tau_1(Y_1, Y_2, Y_3), \tau_2(Y_1, Y_2, Y_3))$ , where  $Y_1, Y_2$  and  $Y_3$  are independent exponential random variables with respective parameters  $\lambda_1, \lambda_2$  and  $\lambda_{12}$ . It is immediate from Theorem 4.1 of Block and Savits (1977) that  $(Y_1, Y_2, Y_3)$  satisfies (4.3.1). Hence from the property (P7) of Block and Savits (1977),  $(X_1, X_2)$  satisfies (4.3.1). However, we have seen already that the joint distribution of  $(X_1, X_2)$  is not MNBU-III.

#### 4.4. Shock Models Leading to MNBU

Consider  $p$  devices which are subjected to shocks occurring randomly in time as events in a general counting process  $N = \{N(t): t \geq 0\}$ , not necessarily Poisson. Let  $\bar{P}_{k_1, \dots, k_p}$  denote the probability of surviving  $k_1, \dots, k_p$  shocks by the component 1, ...,  $p$  respectively. For simplicity, we shall confine ourselves to the bivariate case (i. e.  $p = 2$ ) in this section, although all the main results have direct multivariate generalizations.



We assume  $\bar{P}_{k_1, k_2}$  is nonincreasing in each argument when the other is held fixed and  $\bar{P}_{0,0} = 1$ . Now the survival function can be written as

$$\begin{aligned}
 \bar{H}(t_1, t_2) &= \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} P(N(t_1) = k_1, N(t_2) - N(t_1) = k_2) \\
 &\quad \bar{P}_{k_1, k_1+k_2} \quad \text{for } t_1 < t_2 \\
 &= \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} P(N(t_2) = k_2, N(t_1) - N(t_2) = k_1) \\
 &\quad \bar{P}_{k_1+k_2, k_1} \quad \text{for } t_2 < t_1 \\
 \bar{H}(t, t) &= \sum_{k=0}^{\infty} P(N(t) = k) \bar{P}_{k, k}. \tag{4.4.1}
 \end{aligned}$$

Such shock models are considered in Marshall and Shaked (1979).

These are bivariate generalizations of the shock model considered by Esary, Marshall and Proschan (1973), Hameed and Proschan (1973, 1975) and Block and Savits (1978).

We first generalize the Block-Savits (1978) Theorem 3.1 to the MNBU-IV case.

Theorem 2. Let

- (i)  $\bar{P}_{k_1+k_2, \ell_1+\ell_2} \leq \bar{P}_{k_1, \ell_1} \bar{P}_{k_2, \ell_2}$  for all  $k_i \geq 0, \ell_i \geq 0$  ( $i = 1, 2$ );
- (ii)  $\bar{P}_{s_1, s_2} \rightarrow 0$  when  $\max(s_1, s_2) \rightarrow \infty$ ; and

(iii)  $P(N(x+y) \leq j+k, N(x) = k) \leq P(N(y) \leq j) P(N(x) = k)$  for all  $x \geq 0, y \geq 0, j = 0, 1, \dots$ , and  $k = 0, 1, 2, \dots$ .

Then  $\overline{H}$  defined in (4.4.1) is MNBU-IV.

Proof.

$$\begin{aligned}
 \overline{H}(t+\Delta, t+\Delta) &= \sum_{\ell=0}^{\infty} P(N(t+\Delta) = \ell) \overline{P}_{\ell, \ell} \\
 &= \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} P(N(t+\Delta) = \ell, N(t) = k) \overline{P}_{\ell, \ell} \\
 &= \sum_{k=0}^{\infty} \sum_{\ell=k}^{\infty} P(N(t+\Delta) = \ell, N(t) = k) \overline{P}_{\ell, \ell} \\
 &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} P(N(t+\Delta) = j+k, N(t) = k) \overline{P}_{j+k, j+k} \\
 &\leq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} P(N(t+\Delta) = j+k, N(t) = k) \overline{P}_{j, j} \overline{P}_{k, k} \quad (\text{from (i)}) \\
 &= \sum_{k=0}^{\infty} \overline{P}_{k, k} \sum_{j=0}^{\infty} P(N(t+\Delta) = j+k, N(t) = k) \\
 &\quad \sum_{\ell=j}^{\infty} (\overline{P}_{\ell, \ell} - \overline{P}_{\ell+1, \ell+1}) \quad (\text{using (ii)}) \\
 &= \sum_{k=0}^{\infty} \overline{P}_{k, k} \sum_{\ell=0}^{\infty} (\overline{P}_{\ell, \ell} - \overline{P}_{\ell+1, \ell+1}) \sum_{j=0}^{\ell} P(N(t+\Delta) = j+k, N(t) = k) \\
 &\leq \sum_{k=0}^{\infty} \overline{P}_{k, k} \sum_{\ell=0}^{\infty} (\overline{P}_{\ell, \ell} - \overline{P}_{\ell+1, \ell+1}) P(N(\Delta) \leq \ell) P(N(t) = k) \\
 &\quad (\text{using (iii)})
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} \bar{P}_{k,k} P(N(t)=k) \sum_{\ell=0}^{\infty} (\bar{P}_{\ell,\ell} - \bar{P}_{\ell+1,\ell+1}) \sum_{j=0}^{\ell} P(N(\Delta)=j) \\
&= \sum_{k=0}^{\infty} \bar{P}_{k,k} P(N(t)=k) \sum_{j=0}^{\infty} P(N(\Delta)=j) \sum_{\ell=j}^{\infty} (\bar{P}_{\ell,\ell} - \bar{P}_{\ell+1,\ell+1}) \\
&= \bar{H}(t,t) \sum_{j=0}^{\infty} P(N(\Delta)=j) \bar{P}_{j,j} = \bar{H}(t,t) \bar{H}(\Delta,\Delta).
\end{aligned}$$

Remark 9. The condition (iii) of Theorem 2 is satisfied when the counting process  $N$  is stationary and independent increments.

We have been unable to prove the MNBU-II or MNBU-III property of MNBU bivariate survival function  $\bar{H}$  under the conditions of Theorem 2. However, the following stronger theorem holds if (iii) of Theorem 2 is replaced by the stronger condition (iv).

Theorem 3. If (i) and (ii) of Theorem 2 hold, while (iii) of Theorem 2 is replaced by the stronger condition (iv) the counting process  $N$  is stationary and independent increments, then the survival function  $\bar{H}$  defined in (4.4.1) is MNBU-II.

Proof. For  $t_1 \leq t_2$ ,  $\Delta_1 \leq \Delta_2$

$$\begin{aligned}
\bar{H}(t_1 + \Delta_1, t_2 + \Delta_2) &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} P(N(t_1 + \Delta_1) = k_1, N(t_2 + \Delta_2) \\
&\quad - N(t_1 + \Delta_1) = k_2) \bar{P}_{k_1, k_1 + k_2}
\end{aligned}$$

$$= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} P(N(t_1 + \Delta_1) = k_1, N(t_1) = j_1)$$

$$P(N(t_2 + \Delta_2 - t_1 - \Delta_1) = k_2, N(t_2 - t_1) = j_2)$$

$$\overline{P}_{k_1, k_1 + k_2}$$

$$= \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{k_1=j_1}^{\infty} \sum_{k_2=j_2}^{\infty} P(N(t_1) = j_1) P(N(\Delta_1) = k_1 - j_1)$$

$$P(N(t_2 - t_1) = j_2) P(N(\Delta_2 - \Delta_1) = k_2 - j_2)$$

$$\overline{P}_{k_1, k_1 + k_2} \quad (\text{using (iv)})$$

$$= \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} P(N(t_1) = j_1) P(N(\Delta_1) = k_1)$$

$$P(N(t_2 - t_1) = j_2) P(N(\Delta_2 - \Delta_1) = k_1)$$

$$\overline{P}_{k_1 + j_1, k_1 + k_2 + j_1 + j_2}$$

$$\leq \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} P(N(t_1) = j_1, N(t_2) = j_1 + j_2)$$

$$P(N(\Delta_1) = k_1, N(\Delta_2) = k_1 + k_2)$$

$$\overline{P}_{k_1, k_1 + k_2} \overline{P}_{j_1, j_1 + j_2} \quad (\text{using (i)})$$

$$= \overline{H}(t_1, t_2) \overline{H}(\Delta_1, \Delta_2).$$

We do not know whether the conditions of Theorem 3 are sufficient to prove MNBU-I. Note, however that (iv) is satisfied for Poisson processes, generalized Poisson processes and compound Poisson processes. One set of sufficient conditions for (i) to hold is given in Marshall and Shaked (1979).

We can also assume that the devices are subject to shocks which come from different sources. In such situations it is more meaningful to use a multidimensional counting process. In the special bivariate case we define the counting process  $N$  to be  $N = \{(N_1(t), N_2(s)), t \geq 0, s \geq 0\}$ . MNBU properties of survival functions when shocks are governed by a bivariate Poisson process are currently under investigation.

Finally, we consider the situation when shocks are governed by a nonhomogeneous Poisson process.

Let

$$\begin{aligned} \overline{H}^*(t_1, t_2) &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \exp[-\Lambda(t_1)] \frac{\Lambda^{k_1}(t_1)}{k_1!} \\ &\quad \exp[-\Lambda(t_2 - t_1)] \frac{(\Lambda(t_2 - t_1))^{k_2}}{k_2!} \\ \overline{P}_{k_1, k_1+k_2} &\quad \text{for } 0 \leq t_1 \leq t_2 \end{aligned}$$

$$\begin{aligned}
&= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \exp[-\wedge(t_2)] \frac{\wedge^{k_2}(t_2)}{k_2!} \\
&\quad \exp[-\wedge(t_1 - t_2)] \frac{(\wedge(t_1 - t_2))^{k_1}}{k_1!} \\
&\quad \overline{P}_{k_1+k_2, k_1} \text{ for } 0 \leq t_2 \leq t_1,
\end{aligned}$$

where  $\wedge$  satisfies the super additivity condition  $\wedge(t + \Delta) \geq \wedge(t) + \wedge(\Delta)$  for all  $t \geq 0$ ,  $\Delta \geq 0$ . We shall show now that  $\overline{H}^*$  is MNBU-IV.

To see this, note that when writing

$$\overline{H}(t, t) = \sum_{k=0}^{\infty} \exp(-t) \frac{t^k}{k!} \overline{P}_{k, k},$$

one gets

$$\begin{aligned}
-\log \overline{H}^*(t+x, t+x) &= -\log \overline{H}(\wedge(t+x), \wedge(t+x)) \\
&\geq -\log \overline{H}(\wedge(t) + \wedge(x), \wedge(t) + \wedge(x)) \\
&\quad (\text{using super additivity of } \wedge) \\
&\geq -\log [\overline{H}(\wedge(t), \wedge(t))] [\overline{H}(\wedge(x), \wedge(x))] \\
&= -\log [\overline{H}^*(t, t) \overline{H}^*(x, x)].
\end{aligned}$$

Hence,  $\overline{H}^*(t+x, t+x) \leq \overline{H}^*(t, t) \overline{H}^*(x, x)$  for all  $t \geq 0$ ,  $x \geq 0$ , i.e., it is MNBU-IV.

#### 4.5. Definitions and Example of MNBUE

The notations  $H_1$  and  $\overline{H}_p$  remain the same as in the earlier

sections. In the univariate case a nonnegative random variable  $X_1$  is said to have a NBUE (NWUE) distribution if

$$\int_t^\infty \bar{H}_1(x) dx \leq (\geq) \bar{H}_1(t) \int_0^\infty \bar{H}_1(x) dx, \quad (4.5.1)$$

for all  $t \geq 0$ , where it is assumed that  $\int_0^\infty \bar{H}_1(x) dx < \infty$ .

With the alternate representation

$$E[X_1 - t | X_1 > t] \leq (\geq) E X_1 = E(X_1 - 0 | X_1 > 0) \quad (4.5.2)$$

of (4.5.1) it is easy to see that the condition is equivalent to saying that the conditional mean residual lifetime of a unit which has survived up to time  $t$  is less (greater than) or equal to the mean lifetime of a new unit.

Various multivariate generalizations of (4.5.1) were considered by Buchanan and Singpurwalla (1977) of which we take the following three definitions of MNBUE (MNWUE).

Definitions.  $H_p$  is said to be

(i) MNBUE-I (MNWUE-I) if

$$\int_{t_p}^\infty \dots \int_{t_1}^\infty \bar{H}_p(x_1, \dots, x_p) dx_1, \dots, dx_p \leq (\geq) \bar{H}_p(t_1, \dots, t_p) \int_0^\infty \dots \int_0^\infty \bar{H}_p(x_1, \dots, x_p) dx_1, \dots, dx_p \quad (4.5.3)$$

for all  $t_i \geq 0$  ( $1 \leq i \leq p$ ), and similar inequalities are assumed to hold for all subsets of random variables, where it is assumed that

integrals of the type  $\int_0^\infty \dots \int_0^\infty \overline{H}_p(x_1, \dots, x_p) dx_1, \dots, dx_p$  are all finite;

(ii) MNBUE-II (MNWUE-II) if

$$\int_t^\infty \dots \int_t^\infty \overline{H}_p(x_1, \dots, x_p) dx_1, \dots, dx_p \leq (\geq) \overline{H}_p(t, \dots, t)$$

$$\int_0^\infty \dots \int_0^\infty \overline{H}_p(x_1, \dots, x_p) dx_1, \dots, dx_p \quad (4.5.4)$$

for all  $t \geq 0$ , and similar inequalities hold for all subsets of random variables, where  $\int_0^\infty \dots \int_0^\infty \overline{H}_p(x_1, \dots, x_p) dx_1, \dots, dx_p$  similar integrals are all assumed to be finite;

(iii) MNBUE-III (MNWUE-III) if

$$\int_t^\infty \overline{H}_p(x, \dots, x) dx \leq \overline{H}_p(t, \dots, t) \int_0^\infty \overline{H}_p(x, \dots, x) dx \quad (4.5.5)$$

for all  $t \geq 0$ , and similar inequalities hold for all subsets of variables, where it is assumed that  $\int_0^\infty P(X_i > x) dx < \infty$  for all  $i = 1, \dots, p$ .

We now give physical interpretation to these three definitions.

First note that using the Fubini Theorem:

$$\begin{aligned} & \int_t^\infty \dots \int_t^\infty \prod_{i=1}^p (x_i - t_i) dH_p(x_1, \dots, x_p) \\ &= \int_0^\infty \dots \int_0^\infty \prod_{i=1}^p x_i dH_p(x_1 + t_1, \dots, x_p + t_p) \\ &= \int_0^\infty \dots \int_0^\infty \int_0^{x_p} \dots \int_0^{x_1} dy_1, \dots, dy_p dH_p(x_1 + t_1, \dots, x_p + t_p) \end{aligned}$$



$$\begin{aligned}
&= \int_0^\infty \cdots \int_0^\infty \int_{y_p}^\infty \cdots \int_{y_1}^\infty dH_p(x_1 + t_1, \dots, x_p + t_p) dy_1, \dots, dy_p \\
&= \int_0^\infty \cdots \int_0^\infty \bar{H}_p(y_1 + t_1, \dots, y_p + t_p) dy_1, \dots, dy_p \\
&= \int_{t_1}^\infty \cdots \int_{t_p}^\infty \bar{H}_p(y_1, \dots, y_p) dy_1, \dots, dy_p. \tag{4.5.6}
\end{aligned}$$

Hence, from (4.5.6),

$$\begin{aligned}
&E \left[ \prod_{i=1}^p (X_i - t_i) \mid X_1 > t_1, \dots, X_p > t_p \right] \\
&= \frac{\int_{t_p}^\infty \cdots \int_{t_1}^\infty \prod_{i=1}^p (x_i - t_i) dH(x_1, \dots, x_p)}{\bar{H}_p(t_1, \dots, t_p)} \\
&= \int_{t_p}^\infty \cdots \int_{t_1}^\infty \bar{H}_p(x_1, \dots, x_p) dx_1, \dots, dx_p / \bar{H}_p(t_1, \dots, t_p). \tag{4.5.7}
\end{aligned}$$

Now, using (4.5.7) it follows that (4.5.3) is equivalent to the statement that the conditional mean residual product lifetime of the components of a unit with the components surviving ages  $t_1, \dots, t_p$  respectively is less (greater) than or equal to the mean product lifetime of the components of a new unit. Similarly, the definition (ii) is equivalent to the statement that the conditional mean residual product lifetime of the components of a unit when all the components have survived a certain time  $t$  is less (greater) than or equal to the mean product lifetime of the components of a new unit. The definition (iii)

is equivalent to the statement that a multivariate distribution is NBUE (NWUE) if the minimum of the components has a univariate NBUE (NWUE) distribution.

It is claimed in Buchanan and Singpurwalla (1977) that

$$\text{MNBUE-I} \Rightarrow \text{MNBUE-II} \Rightarrow \text{MNBUE-III}. \quad (4.5.8)$$

It is trivial to check that  $\text{MNBUE-I} \Rightarrow \text{MNBUE-II}$ . However, contrary to the claim made by these authors, the following example shows that  $\text{MNBUE-I} \not\Rightarrow \text{MNBUE-III}$  (so that  $\text{MNBUE-II} \not\Rightarrow \text{MNBUE-III}$ ).

Example 13. Let  $X_1$  and  $X_2$  be independent and identically distributed with common survival function

$$\begin{aligned} \overline{F}(x) &= 1 & \text{if } 0 \leq x < 3 \\ &= \frac{1}{4} & \text{if } 3 \leq x < 7 \\ &= 0 & \text{if } x \geq 7 \end{aligned} \quad (4.5.9)$$

Then, for  $0 \leq t < 3$ ,  $\int_t^\infty \overline{F}(x) dx = (3-t) + 1 = 4-t$  so that  $\int_0^7 \overline{F}(x) dx = 4$ . Since  $\overline{F}(t) = 1$  for  $0 \leq t < 3$ , (4.5.1) holds for  $0 \leq t < 3$ . For  $3 \leq t < 7$ ,  $\int_t^7 \overline{F}(x) dx = \frac{1}{4}(7-t) \leq 1$  for  $3 \leq t < 7$ . For  $t \geq 7$ , (4.5.1) is true both sides being equal to zero. Thus, the common distribution of  $X_1$  and  $X_2$  is NBUE. Since  $X_1$  and  $X_2$  are independent and identically distributed, their joint survival function must satisfy the MNBUE-I property. However, since

$P(\min(X_1, X_2) > x) = \overline{F}^2(x), \int_0^7 \overline{F}^2(x) dx = 3.25, \int_3^7 \overline{F}^2(x) dx = \frac{1}{4}$   
 and  $\overline{F}^2(3) = \frac{1}{16}$  it follows that  $\int_3^7 \overline{F}^2(x) dx > \overline{F}^2(3) \int_0^7 \overline{F}^2(x) dx$   
 so that  $\min(X_1, X_2)$  does not have a NBUE distribution

Next we show that MNBUE-III  $\not\equiv$  MNBUE-II. To see this, consider once again the Example 3 with  $\lambda_1 = \lambda_2 = 0, \lambda_3 > 0$  and  $\lambda_4 > 0$ . In this case  $\min(X_1, X_2)$  is exponential with parameter  $\max(\lambda_3, \lambda_4)$ ,  $X_1$  is exponential with parameter  $\lambda_3$ ,  $X_2$  is exponential with parameter  $\lambda_4$ , and hence each has a NBU as well as a NWU distribution. However, for  $t \geq 0, \lambda_3 > \lambda_4$ ,

$$\begin{aligned} \int_t^\infty \int_t^\infty \overline{H}(x_1, x_2) dx_1 dx_2 &= \int_t^\infty \int_{\max(\frac{\lambda_4}{\lambda_3} x_2, t)}^\infty \lambda_4 \exp(-\lambda_3 x_1) dx_1 dx_2 \\ &+ \int_t^\infty \int_{\max(\frac{\lambda_3}{\lambda_4} x_1, t)}^\infty \lambda_3 \exp(-\lambda_4 x_2) dx_2 dx_1 \\ &= \lambda_3^{-1} \int_t^\infty \exp(-\lambda_3 \max(\frac{\lambda_4}{\lambda_3} x_2, t)) dx_2 \\ &+ \lambda_4^{-1} \int_t^\infty \exp(-\lambda_4 \max(\frac{\lambda_3}{\lambda_4} x_1, t)) dx_1 \\ &= \lambda_3^{-1} \int_t^\infty \exp(-\max(\lambda_4 x_2, \lambda_3 t)) dx_2 \\ &+ \lambda_4^{-1} \int_t^\infty \exp(-\max(\lambda_3 x_1, \lambda_4 t)) dx_1 \end{aligned}$$

$$\begin{aligned}
&= \lambda_3^{-1} \left[ \int_t^{\frac{\lambda_3}{\lambda_4}t} \exp(-\lambda_3 t) dx_2 + \int_{\frac{(\frac{\lambda_3}{\lambda_4})t}^{\infty} \exp(-\lambda_4 x_2) dx_2 \right] \\
&+ \lambda_4^{-1} \int_t^{\infty} \exp(-\lambda_3 x_1) dx_1 \\
&= \lambda_3^{-1} \exp(-\lambda_3 t) \left( \frac{\lambda_3}{\lambda_4} - 1 \right) t + (\lambda_3 \lambda_4)^{-1} \exp(-\lambda_3 t) \\
&+ \lambda_4^{-1} \lambda_3^{-1} \exp(-\lambda_3 t) \\
&= \exp(-\lambda_3 t) \left[ (\lambda_4^{-1} - \lambda_3^{-1}) t + 2 (\lambda_3 \lambda_4)^{-1} \right]; \tag{4.5.10}
\end{aligned}$$

$$\begin{aligned}
\overline{H}(t, t) &= \exp(-\lambda_3 t) \text{ and } \int_0^{\infty} \int_0^{\infty} \overline{H}(x_1, x_2) dx_1 dx_2 \\
&= 2 (\lambda_3 \lambda_4)^{-1} \tag{4.5.11}
\end{aligned}$$

so that from (4.5.10) and (4.5.11),

$$\int_t^{\infty} \int_t^{\infty} \overline{H}(x_1, x_2) dx_1 dx_2 \geq \overline{H}(t, t) \int_0^{\infty} \int_0^{\infty} \overline{H}(x_1, x_2) dx_1 dx_2$$

for all  $t \geq 0$  so that  $(X_1, X_2)$  has a joint MNWUE-II survival function rather than a MNBUE-III survival function.

It is known in the univariate case that  $\text{NBU} \Rightarrow \text{NBUE}$ . The MNBU chain implications are clearly verified in Section 2. It is immediate from definitions that  $\text{MNBU-I} \Rightarrow \text{MNBUE-I}$ ,  $\text{MNBU-III} \Rightarrow \text{MNBUE-II}$  and  $\text{MNBU-IV} \Rightarrow \text{MNBUE-III}$ . Thus Example 13 also illustrates that  $\text{MNBUE-I} \not\Rightarrow \text{MNBU-IV}$  (and hence does not imply

anyone of MNBU-I, MNBU-II or MNBU-III). Example 3 shows that  $\text{MNBUE-III} \not\Rightarrow \text{MNBUE-II}$ . To show that  $\text{MNBUE-III} \not\Rightarrow \text{MNBU-IV}$ , consider two identically independent distributed random variables  $Y_1$  and  $Y_2$  with common distribution function  $1 - \frac{1}{\bar{F}^2}(x)$  where  $\bar{F}(x)$  is defined in (4.5.8). Then  $\min(Y_1, Y_2)$  has a NBUE distribution, but not a NBU distribution.

Buchanan and Singpurwalla (1977) also claim that  $\text{MNWUE-I} \Rightarrow \text{MNWUE-II} \Rightarrow \text{MNWUE-III}$ . That  $\text{MNWUE-I} \Rightarrow \text{MNWUE-II}$  is immediate from the definition. We have not been, however, able to verify that  $\text{MNWUE-II} \Rightarrow \text{MNWUE-III}$ .

#### 4.6. Closure Properties of MNBUE

In this section, we prove certain closure properties of MNBUE distributions comparable to those proved in Section 3.

Let  $\beta_1, \beta_2$  and  $\beta_3$  denote the classes of life distributions satisfying the definitions (i), (ii) and (iii) respectively of MNBUE. Then we have the following theorem.

##### Theorem 4.

- (Q1) If  $(T_1, \dots, T_m) \in \beta_j$ , any subset of  $(T_1, \dots, T_m) \in \beta_j$  ( $1 \leq j \leq 3$ );
- (Q2) if  $(T_1, \dots, T_m) \in \beta_j$ ,  $(T'_1, \dots, T'_n) \in \beta_j$ , and  $(T_1, \dots, T_m)$  and  $(T'_1, \dots, T'_n)$  are independently distributed, then,  $(T_1, \dots, T_m, T'_1, \dots, T'_n) \in \beta_j$  ( $1 \leq j \leq 3$ );

(Q3) if  $(T_1, \dots, T_m) \in \beta_1$ , then  $(C_1 T_1, \dots, C_m T_m) \in \beta_1$  for all  $C_i > 0$  ( $1 \leq i \leq m$ ). If  $(T_1, \dots, T_m) \in \beta_j$ , then  $(C T_1, \dots, C T_m) \in \beta_j$  for all  $C > 0$  ( $j = 2, 3$ );

(Q4)  $\beta_j$  ( $1 \leq j \leq 3$ ) is closed under convolution (whenever the operation is meaningful).

Proof.

(Q1) This property follows immediately from the definitions.

(Q2) Note that if  $(T_1, \dots, T_m) \in \beta_1$  and  $(T'_1, \dots, T'_n) \in \beta_1$ , then using the independence of  $(T_1, \dots, T_m, T'_1, \dots, T'_n)$ , for  $x_i \geq 0$ ,  $x'_i \geq 0$ ,  $t_i \geq 0$ ,  $t'_i \geq 0$ ,

$$\begin{aligned}
 & \int_0^\infty \dots \int_0^\infty P(T_1 > x_1 + t_1, \dots, T_m > x_m + t_m, T'_1 > x'_1 + t'_1, \dots, T'_n \\
 & \quad > x'_n + t'_n) dx_1 \dots dx_m dx'_1 \dots dx'_n \\
 &= \left[ \int_0^\infty \dots \int_0^\infty P(T_1 > x_1 + t_1, \dots, T_m > x_m + t_m) dx_1 \dots dx_m \right] \\
 & \quad \left[ \int_0^\infty \dots \int_0^\infty P(T'_1 > x'_1 + t'_1, \dots, T'_n > x'_n + t'_n) dx'_1 \dots dx'_n \right] \\
 &\leq P\left(\bigcap_{i=1}^m (T_i > t_i)\right) \int_0^\infty \dots \int_0^\infty P(T_1 > x_1, \dots, T_m > x_m) dx_1 \dots \\
 & \quad dx_m \times P\left(\bigcap_{i=1}^n (T'_i > t'_i)\right) \int_0^\infty \dots \int_0^\infty P(T'_1 > x'_1, \dots, \\
 & \quad T'_n > x'_n) dx'_1 \dots dx'_n
 \end{aligned}$$

$$\begin{aligned}
&= P(T_1 > t_1, \dots, T_m > t_m, T'_1 > t'_1, \dots, T'_n > t'_n) \\
&\quad \int_0^\infty \dots \int_0^\infty P(T_1 > x_1, \dots, T_m > x_m, T'_1 > x'_1, \dots, T'_n > x'_n) \\
&\quad \quad dx_1 \dots dx_m dx'_1 \dots dx'_n.
\end{aligned}$$

Similar proofs work for  $\beta_2$  and  $\beta_3$ .

(Q3) If  $(T_1, \dots, T_m) \in \beta_1$ ,  $C_i > 0$  ( $1 \leq i \leq m$ ),

$$\begin{aligned}
&\int_0^\infty \dots \int_0^\infty P(C_1 T_1 > x_1 + t_1, \dots, C_m T_m > x_m + t_m) dx_1 \dots dx_m \\
&= \int_0^\infty \dots \int_0^\infty P(T_1 > \frac{x_1 + t_1}{C_1}, \dots, T_m > \frac{x_m + t_m}{C_m}) dx_1 \dots dx_m \\
&\leq P(T_1 > \frac{t_1}{C_1}, \dots, T_m > \frac{t_m}{C_m}) \int_0^\infty \dots \int_0^\infty P(T_1 > \frac{x_1}{C_1}, \dots, T_m > \frac{x_m}{C_m}) \\
&\quad \quad \quad dx_1 \dots dx_m \\
&= P(C_1 T_1 > t_1, \dots, C_m T_m > t_m) \int_0^\infty \dots \int_0^\infty P(C_1 T_1 > x_1, \dots, \\
&\quad \quad \quad C_m T_m > x_m) dx_1 \dots dx_m.
\end{aligned}$$

Similar proofs work for  $\beta_2$  and  $\beta_3$  when  $C_1 = C_2 = \dots = C_m = C > 0$ .

(Q4) To prove this property for  $\beta_1$ , we need the following multivariate extension of Lemma 3.4 (ii) in Block and Savits (1978).

Lemma 2.  $H_p$  is MNBUE-I if and only if

$$\int_0^\infty \dots \int_0^\infty g(z_1, \dots, z_p) \overline{H}_p(z_1, \dots, z_p) dz_1 \dots dz_p$$

$$\leq (\int_0^\infty \dots \int_0^\infty \overline{H}_p(z_1, \dots, z_p) dz_1 \dots dz_p) \\ (\int_0^\infty \dots \int_0^\infty g(z_1, \dots, z_p) dH(z_1, \dots, z_p))$$

for every nonnegative nondecreasing real  $g$  on  $(0, \infty)^P$ .

Proof. The sufficiency part of the theorem follows by taking

$$g(z_1, \dots, z_p) = \prod_{i=1}^P I_{[z_i > t_i]}. \quad \text{To prove necessity, first note that}$$

the integral inequality is valid for functions of the form

$$g(z_1, \dots, z_p) = \prod_{i=1}^P I_{[z_i \in A_i]} \quad \text{where } A_i \text{ is either } (t_i, \infty) \text{ or } [t_i, \infty).$$

Since, any nonnegative nondecreasing function can be written as the nondecreasing limit of nonnegative linear combinations of such functions, the result follows.

To prove the closure property  $(Q_4)$  under  $\beta_1$ , we now proceed as follows.

Suppose  $\underset{\sim}{X} = (X_1, \dots, X_p)$  and  $\underset{\sim}{Y} = (Y_1, \dots, Y_p)$  are independent random vectors with respective distribution functions  $F_p$  and  $G_p$  and corresponding survival functions  $\overline{F}_p$  and  $\overline{G}_p$ . Let  $H_p$  and  $\overline{H}_p$  denote the distribution function and the survival function of  $\underset{\sim}{Z} = \underset{\sim}{X} + \underset{\sim}{Y}$ . Then, writing

$$\int_0^\infty \dots \int_0^\infty g(z_1, \dots, z_p) \overline{H}_p(z_1, \dots, z_p) dz_1 \dots dz_p \\ = \int_0^\infty \dots \int_0^\infty \int_0^{z_p} \dots \int_0^{z_1} g(u_1, \dots, u_p) du_1 \dots du_p dH(z_1, \dots, z_p)$$



$$\begin{aligned}
&= \int_0^\infty \dots \int_0^\infty \int_0^{x_p+y_p} \dots \int_0^{x_1+y_1} g(u_1, \dots, u_p) du_1 \dots du_p \\
&\quad dF(x_1, \dots, x_p) dG(y_1, \dots, y_p) \\
&= \int_0^\infty \dots \int_0^\infty \int_0^{x_p} \dots \int_0^{x_1} g(u_1, \dots, u_p) du_p \dots du_p \\
&\quad dF(x_1, \dots, x_p) dG(y_1, \dots, y_p) \\
&\quad + \int_0^\infty \dots \int_0^\infty \int_0^{y_p} \dots \int_0^{y_1} g(x_1+u_1, \dots, x_p+u_p) du_1 \dots du_p \\
&\quad dG(y_1, \dots, y_p) dF(x_1, \dots, x_p) \\
&= \int_0^\infty \dots \int_0^\infty \left[ \int_0^\infty \dots \int_0^\infty g(u_1, \dots, u_p) \overline{F}_p(u_1, \dots, u_p) du_1 \dots \right. \\
&\quad \left. du_p \right] dG(y_1, \dots, y_p) \\
&\quad + \int_0^\infty \dots \int_0^\infty \left[ \int_0^\infty \dots \int_0^\infty g(x_1+u_1, \dots, x_p+u_p) \overline{G}_p(u_1, \dots, u_p) \right. \\
&\quad \left. du_1 \dots du_p \right] dF(x_1, \dots, x_p) \\
&\leq \left[ \int_0^\infty \dots \int_0^\infty \overline{F}_p(u_1, \dots, u_p) du_1 \dots du_p \right] \left[ \int_0^\infty \dots \int_0^\infty \right. \\
&\quad \left. \left( \int_0^\infty \dots \int_0^\infty g(x_1, \dots, x_p) dF(x_1, \dots, x_p) \right) \right. \\
&\quad \left. dG(y_1, \dots, y_p) \right] \\
&\quad + \left[ \int_0^\infty \dots \int_0^\infty \overline{G}_p(u_1, \dots, u_p) du_1 \dots du_p \right] \left[ \int_0^\infty \dots \int_0^\infty \right. \\
&\quad \left. \left( \int_0^\infty \dots \int_0^\infty g(x_1+y_1, \dots, x_p+y_p) dG(y_1, \dots, y_p) \right) \right. \\
&\quad \left. dF(x_1, \dots, x_p) \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \left[ \int_0^\infty \dots \int_0^\infty \{ \overline{F}_p(u_1, \dots, u_p) + \overline{G}_p(u_1, \dots, u_p) \} du_p \dots du_p \right] \\
&\quad \times \int_0^\infty \dots \int_0^\infty g(x_1 + y_1, \dots, x_p + y_p) dF(x_1, \dots, x_p) \\
&\quad \quad \quad dG(y_1, \dots, y_p) \\
&= E \left( \prod_{i=1}^p X_i + \prod_{i=1}^p Y_i \right) \int_0^\infty \dots \int_0^\infty g(z_1, \dots, z_p) dH(z_1, \dots, z_p) \\
&\leq E \left( \prod_{i=1}^p Z_i \right) \int_0^\infty \dots \int_0^\infty g(z_1, \dots, z_p) dH(z_1, \dots, z_p) \\
&\quad \quad \quad \left( \text{since } \prod_{i=1}^p X_i + \prod_{i=1}^p Y_i \leq \prod_{i=1}^{St p} (X_i + Y_i) = \prod_{i=1}^p Z_i \right) \\
&= \left[ \int_0^\infty \dots \int_0^\infty \overline{H}_p(z_1, \dots, z_p) dz_1 \dots dz_p \right] \left[ \int_0^\infty \dots \int_0^\infty \right. \\
&\quad \quad \quad \left. g(z_1, \dots, z_p) dH(z_1, \dots, z_p) \right].
\end{aligned}$$

Similar proof works for  $\beta_2$  and  $\beta_3$ .

Remark 10. Since it is known in the univariate case that NBUE is not closed under the formation of coherent system, the same cannot be expected for MNBUE under any definition.

Remark 11. To prove that MNBUE is closed under limits in distribution, we need an extra condition to guarantee the application dominated convergence theorem.

Theorem 5. Let  $\{(T_{1_k}, \dots, T_{m_k}), k \geq 1\}$  be a sequence of MNBUE random vectors belonging to  $\beta_j$  for each  $k$ . If  $(T_{1_k}, \dots, T_{m_k}) \rightarrow$

$(T_1, \dots, T_k)$  weakly as  $k \rightarrow \infty$  and  $(T_{1_k}, \dots, T_{m_k}) \leq^{St} (S_1, \dots, S_k)$  for all  $m \geq m_0$  where  $E(\prod_{i=1}^k S_i) < \infty$ , then  $(T_1, \dots, T_k) \in \beta_j$  for each  $j$ .

Proof. Use the appropriate definition of MNBUE and the dominated convergence theorem.

#### 4.7. Shock Models Leading to MNBUE

Consider once again the same setup as Section 4. Shocks are assumed to be governed by a general counting process  $N = \{N(t): t \geq 0\}$  not necessarily Poisson. For simplicity, attention restricted only to the bivariate case.

Assume the survival function  $\bar{H}_2(t_1, t_2)$  is defined the same way as (4.4.1). The first theorem in this section is a multivariate generalization of Theorem 2.1 of Block and Savits (1978) dealing with MNBUE survival.

Let

$$a_{k_1, k_2}(t_1, t_2) = P(N(t_1) = k_1, N(t_2) = k_2)$$

and

$$A_{k_1, k_2} = \int_0^\infty \int_0^\infty a_{k_1, k_2}(x_1, x_2) dx_1 dx_2.$$

Theorem 6.  $H_2$  is MNBUE-I if

$$(i) \quad \bar{P}_{k, \ell} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \bar{P}_{k_1, k_2} A_{k_1, k_2} \geq \sum_{k_1=k}^{\infty} \sum_{k_2=\ell}^{\infty} \bar{P}_{k_1, k_2} A_{k_1, k_2} \quad (4.7.1)$$

and

(ii) for every  $t_i \geq 0$ ,  $k_i = 0, 1, \dots$  ( $i = 1, 2$ ),

$$A_{k_1, k_2} \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} a_{j_1, j_2}(t_1, t_2) \geq \int_{t_1}^{\infty} \int_{t_2}^{\infty} a_{k_1, k_2}(x_1, x_2) dx_2 dx_1 \quad (4.7.2)$$

Proof.

$$\begin{aligned} & \bar{H}(t_1, t_2) \int_0^{\infty} \int_0^{\infty} \bar{H}(x_1, x_2) dx_1 dx_2 \\ &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} a_{k_1, k_2}(t_1, t_2) \bar{P}_{k_1, k_2} \int_0^{\infty} \int_0^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} a_{j_1, j_2} \\ & \quad (x_1, x_2) \bar{P}_{j_1, j_2} dx_1 dx_2 \\ &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} a_{k_1, k_2}(t_1, t_2) \bar{P}_{k_1, k_2} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} A_{j_1, j_2} \bar{P}_{j_1, j_2} \\ &\geq \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} a_{k_1, k_2}(t_1, t_2) \sum_{j_1=k_1}^{\infty} \sum_{j_2=k_2}^{\infty} \bar{P}_{j_1, j_2} A_{j_1, j_2} \quad (\text{using (i)}) \\ &= \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \bar{P}_{j_1, j_2} A_{j_1, j_2} \sum_{k_1=0}^{j_1} \sum_{k_2=0}^{j_2} a_{k_1, k_2}(t_1, t_2) \\ &\geq \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \bar{P}_{j_1, j_2} \int_{t_1}^{\infty} \int_{t_2}^{\infty} a_{j_1, j_2}(x_1, x_2) dx_1 dx_2 \end{aligned}$$

$$= \int_t^\infty \int_t^\infty \overline{H}_2(x_1, x_2) dx_1 dx_2.$$

It is difficult to verify that the conditions of Theorem 6 in general. We found it hard to verify the condition even in the special case when  $N$  is Poisson process with intensity say  $\lambda$ .

In the situation when  $N$  is a Poisson process, we could however directly, that  $H_2$  is MNBUE-II, provided  $\{\overline{P}_{k_1, k_2}, k_1 = 0, 1, \dots, k_2 = 0, 1, 2, \dots\}$  satisfies the discrete NBUE property. This is demonstrated in the following theorem.

Theorem 7. Suppose

$$\overline{P}_{k, k} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \overline{P}_{j_1, j_2} \geq \sum_{j_1=k}^{\infty} \sum_{j_2=k}^{\infty} \overline{P}_{j_1, j_2} \quad (4.7.3)$$

and

$$\overline{P}_{k, k} \sum_{j=0}^{\infty} \overline{P}_{j, j} \geq \sum_{j=k}^{\infty} \overline{P}_{k, k}, \quad (4.7.4)$$

then the joint survival function  $\overline{H}_2$  defined in (4.4.1) with a Poisson counting process  $N$  with intensity  $\lambda$  is MNBUE-II.

Proof. For  $t \geq 0$ ,

$$\int_t^\infty \int_t^\infty \overline{H}_2(x_1, x_2) dx_1 dx_2 = I + II \quad (4.7.5)$$

where

$$I = \int_0^\infty \int_{x_2}^\infty \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty \exp[-\lambda \{(x_2+t) + (x_1-x_2)\}] \\ \frac{(\lambda(x_2+t))^{k_2}}{k_2!} \frac{(\lambda(x_1-x_2))^{k_1}}{k_1!} \bar{P}_{k_1+k_2, k_2} dx_1 dx_2,$$

and

$$II = \int_0^\infty \int_{x_1}^\infty \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty \exp[-\lambda \{(x_1+t) + (x_2-x_1)\}] \\ \frac{(\lambda(x_1+t))^{k_1}}{k_1!} \frac{(\lambda(x_2-x_1))^{k_2}}{k_2!} \bar{P}_{k_1, k_1+k_2} dx_2 dx_1.$$

Now,

$$I = \lambda^{-1} \int_0^\infty \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty \exp[-\lambda(x_2+t)] \\ \frac{(\lambda(x_2+t))^{k_2}}{k_2!} \bar{P}_{k_1+k_2, k_2} dx_2 \\ = \lambda^{-2} \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty \bar{P}_{k_1+k_2, k_2} \left[ \int_{\lambda t}^\infty \frac{\exp(-y) y^{k_2}}{k_2!} dy \right] \\ = \lambda^{-2} \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty \bar{P}_{k_1+k_2, k_2} \sum_{j=0}^{k_2} \exp(-\lambda t) \frac{(\lambda t)^j}{j!}$$

Similarly,

$$II = \lambda^{-2} \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty \bar{P}_{k_1, k_1+k_2} \sum_{j=0}^{k_1} \exp(-\lambda t) \frac{(\lambda t)^j}{j!}.$$

Hence,

$$\begin{aligned}
 I + II = \lambda^{-2} & \left[ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \overline{P}_{k_1, k_2} \sum_{j=0}^{\min(k_1, k_2)} \exp(-\lambda t) \frac{(\lambda t)^j}{j!} \right. \\
 & \left. + \sum_{k=0}^{\infty} \overline{P}_{k, k} \sum_{j=0}^k \exp(-\lambda t) \frac{(\lambda t)^j}{j!} \right]. \quad (4.7.6)
 \end{aligned}$$

From (4.7.5) and (4.7.6) one gets,

$$\int_0^{\infty} \int_0^{\infty} \overline{H}_2(x_1, x_2) dx_1 dx_2 = \lambda^{-2} \left[ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \overline{P}_{k_1, k_2} + \sum_{k=0}^{\infty} \overline{P}_{k, k} \right]. \quad (4.7.7)$$

Since, from (4.4.1),

$$\overline{H}_2(t, t) = \sum_{j=0}^{\infty} \exp(-\lambda t) \frac{(\lambda t)^j}{j!} \overline{P}_{j, j} \quad \text{for } t \geq 0, \quad (4.7.8)$$

from (4.7.6) - (4.7.8) one gets,

$$\begin{aligned}
 & \overline{H}_2(t, t) \int_0^{\infty} \int_0^{\infty} \overline{H}_2(x_1, x_2) dx_1 dx_2 - \int_t^{\infty} \int_t^{\infty} \overline{H}_2(x_1, x_2) \\
 & = \lambda^{-2} \left[ \sum_{j=0}^{\infty} \exp(-\lambda t) \frac{(\lambda t)^j}{j!} \overline{P}_{j, j} \left\{ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \overline{P}_{k_1, k_2} + \sum_{k=0}^{\infty} \overline{P}_{k, k} \right\} \right. \\
 & \quad - \left\{ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \overline{P}_{k_1, k_2} \sum_{j=0}^{\min(k_1, k_2)} \exp(-\lambda t) \frac{(\lambda t)^j}{j!} \right. \\
 & \quad \left. \left. + \sum_{k=0}^{\infty} \overline{P}_{k, k} \sum_{j=0}^k \exp(-\lambda t) \frac{(\lambda t)^j}{j!} \right\} \right]
 \end{aligned}$$

$$\begin{aligned}
&= \lambda^{-2} \left[ \sum_{j=0}^{\infty} \exp(-\lambda t) \frac{(\lambda t)^j}{j!} \left\{ \overline{P}_{j,j} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \overline{P}_{k_1,k_2} \right. \right. \\
&\quad \left. \left. - \sum_{k_1=j}^{\infty} \sum_{k_2=j}^{\infty} \overline{P}_{k_1,k_2} \right\} \right. \\
&\quad \left. + \sum_{j=0}^{\infty} \exp(-\lambda t) \frac{(\lambda t)^j}{j!} \left\{ \overline{P}_{j,j} \sum_{k=0}^{\infty} \overline{P}_{k,k} - \sum_{k=j}^{\infty} \overline{P}_{k,k} \right\} \right] \geq 0,
\end{aligned}$$

using (4.7.3) and (4.7.4).

Finally note that if writing  $a_k(t) = a_{k,k}(t, t)$  and  $A_k = \int_0^{\infty} a_k(x) dx$ ,  
if

$$(i) \quad \overline{P}_{k,k} \sum_{j=0}^{\infty} \overline{P}_{j,j} A_j \geq \sum_{j=k}^{\infty} \overline{P}_{j,j} A_j, \text{ and}$$

$$(ii) \quad A_k \sum_{j=0}^k a_j(t) \geq \int_t^{\infty} a_k(x) dx$$

hold, then following the line of proof of Theorem (2.1) of Block and

Savits (1978), it follows that  $\overline{H}_2$  is MNBUE-III.



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